ENSICAEN - 1A Matériaux et Chimie - FISE

TD2 – Variations de fonctions à plusieurs variables

CORRIGE

Exercice 1. *Intégration d'équations aux dérivées partielles*

Soit f une fonction de deux variables et de classe C^2.

(i) $\frac{\partial f}{\partial x} = 0$ admet comme solutions les fonctions telles que $f(x,y) = \varphi_1(y)$

(ii) $\frac{\partial^2 f}{\partial x^2} = 0$ soit $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 0 => \frac{\partial f}{\partial x} = \varphi_2(y)$ et $\frac{\partial^2 f}{\partial x^2} = 0$ admet comme solutions les fonctions telles que $f(x,y) = x.\varphi_2(y) + \psi_1(y)$

(iii) $\frac{\partial^2 f}{\partial x \partial y} = 0$ soit $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0 => \frac{\partial f}{\partial y} = \varphi_3(y)$ et $\frac{\partial^2 f}{\partial x \partial y} = 0$ admet comme solutions les fonctions telles que $f(x,y) = \varphi_4(y) + \psi_2(x)$ avec $\varphi'_4(y) = \varphi_3(y)$

Exercice 2. *Règle de dérivation en chaîne*

F une fonction de classe C^2(R^2, R). $f(x) = F(3x+1, -x) = F(u(x), v(x))$,
$u(x) = 3x + 1$ et $v(x) = -x$

a) $f'(x) = u'(x).\frac{\partial F}{\partial u}(u,v) + v'(x).\frac{\partial F}{\partial v}(u,v) = 3.\frac{\partial F}{\partial u}(u,v) + (-1).\frac{\partial F}{\partial v}(u,v)$

b) $f''(x) = u'(x).\frac{\partial}{\partial u}\left(3\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right) + v'(x).\frac{\partial}{\partial v}\left(3\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right) = 3.\frac{\partial}{\partial u}\left(3\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right) - \frac{\partial}{\partial v}\left(3\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right)$

$f''(x) = 3.\frac{\partial}{\partial u}\left(3\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right) - \frac{\partial}{\partial v}\left(3\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right) = 9\frac{\partial^2 F(u,v)}{\partial u^2} - 3\frac{\partial^2 F(u,v)}{\partial u\,\partial v} - 3\frac{\partial^2 F(u,v)}{\partial v\,\partial u} + \frac{\partial^2 F(u,v)}{\partial v^2}$ soit

$f''(x) = 3.\frac{\partial}{\partial u}\left(3\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right) - \frac{\partial}{\partial v}\left(3\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right) = 9\frac{\partial^2 F(u,v)}{\partial u^2} - 6\frac{\partial^2 F(u,v)}{\partial u\,\partial v} + \frac{\partial^2 F(u,v)}{\partial v^2}$ car $\frac{\partial^2 F(u,v)}{\partial u\,\partial v} = \frac{\partial^2 F(u,v)}{\partial v\,\partial u}$

d'après le théorème de Schwarz qui s'applique car F est de classe C^2.

Exercice 3. *EDP résolue par changement linéaire de variables*

f et F deux fonctions à valeurs réelles, de classe C ^{2} sur R^2. $f(x,y) = F(u,v)$, où $u = U(x,y) = x + ay$ et $v = V(x,y) = x + by$, avec a et b des constantes réelles.

a) La matrice Jacobienne doit être inversible, son déterminant vaut b-a, il faut donc a ≠ b

b) $\frac{\partial f}{\partial x} = \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial F(u,v)}{\partial u} + \frac{\partial F(u,v)}{\partial v} = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}$

$\frac{\partial f}{\partial y} = \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial y} = a\frac{\partial F(u,v)}{\partial u} + b\frac{\partial F(u,v)}{\partial v} = a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\right)\frac{\partial v}{\partial x} = \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial v \partial u} + \frac{\partial^2 F}{\partial v^2}$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial u}\left(a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}\right)\frac{\partial v}{\partial y} = a\frac{\partial}{\partial u}\left(a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}\right) + b\frac{\partial}{\partial v}\left(a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}\right)$ soit

$\frac{\partial^2 f}{\partial y^2} = a^2\frac{\partial^2 F}{\partial u^2} + ab\frac{\partial F}{\partial u\,\partial v} + ab\frac{\partial F}{\partial v\,\partial u} + b^2\frac{\partial^2 F}{\partial v^2} = a^2\frac{\partial^2 F}{\partial u^2} + 2ab\frac{\partial^2 F}{\partial u \partial v} + b^2\frac{\partial^2 F}{\partial v^2}$

Et $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial u}\left(a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}\right)\frac{\partial v}{\partial x} = \frac{\partial}{\partial u}\left(a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}\right) + \frac{\partial}{\partial v}\left(a\frac{\partial F}{\partial u} + b\frac{\partial F}{\partial v}\right)$ soit

$\frac{\partial^2 f}{\partial x \partial y} = a\frac{\partial^2 F}{\partial u^2} + b\frac{\partial F}{\partial u\,\partial v} + a\frac{\partial F}{\partial v\,\partial u} + b\frac{\partial^2 F}{\partial v^2} = a\frac{\partial^2 F}{\partial u^2} + (a+b)\frac{\partial^2 F}{\partial u \partial v} + b\frac{\partial^2 F}{\partial v^2}$

c) $\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0$ est équivalent à $(1-a)\frac{\partial F}{\partial u} + (1-b)\frac{\partial F}{\partial v} = 0$ soit en choisissant a=1 et b=-1, $2\frac{\partial F}{\partial v} = 0$ qui a pour solution $F(u,v) = \varphi(u)$. Et $f(x,y) = \varphi(x+y)$

d) $\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$ est équivalent à $(1-a^2)\frac{\partial^2 F}{\partial u^2} + 2(1-ab)\frac{\partial^2 F}{\partial u \partial v} + (1-b^2)\frac{\partial^2 F}{\partial v^2} = 0$. En choisissant a=1 et b=-1, on a donc $4\frac{\partial^2 F}{\partial u \partial v} = 0$ ou $\frac{\partial^2 F}{\partial u \partial v} = 0$ qui a pour solution $F(u,v) = \varphi(u) + \psi(v)$.

Et $f(x,y) = \varphi(x+y) + \psi(x-y)$

e) L'équation $\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x\, \partial y} + \frac{\partial^2 f}{\partial y^2} = 0$ est équivalente à

$(1+a^2-2a)\frac{\partial^2 F}{\partial u^2} + 2(1+ab-a-b)\frac{\partial^2 F}{\partial u \partial v} + (1+b^2-2b)\frac{\partial^2 F}{\partial v^2} = 0$ soit

$(1-a)^2\frac{\partial^2 F}{\partial u^2} + 2(1-a)(1-b))\frac{\partial^2 F}{\partial u \partial v} + (1-b)^2\frac{\partial^2 F}{\partial v^2} = 0$. En prenant a=0 et b=1 (u=x et v=x+y), on a $\frac{\partial^2 F}{\partial u^2} = 0$ qui a pour solution $F(u,v) = u.\varphi(v) + \psi(v)$, et donc $f(x,y) = x.\varphi(x+y) + \psi(x+y)$

Exercice 4. *EDP résolue par changement de variables*

Soit f une fonction de deux variables x et y, de classe C^2 sur l'ouvert $U = \{(x,y) \in R^2 / x-y > 0\}$. On résoud $\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y) + 7(x-y)f(x,y) = 0$

a) $f(x,y) = g(u,v)$, où $u = x.y$ et $v = x+y$, et g est une fonction de classe C^2 sur un ouvert.

$\frac{\partial f}{\partial x} = \frac{\partial g(u,v)}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g(u,v)}{\partial v}\frac{\partial v}{\partial x} = y.\frac{\partial g(u,v)}{\partial u} + \frac{\partial g(u,v)}{\partial v} = y.\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}$

$\frac{\partial f}{\partial y} = \frac{\partial g(u,v)}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g(u,v)}{\partial v}\frac{\partial v}{\partial y} = x.\frac{\partial g(u,v)}{\partial u} + \frac{\partial g(u,v)}{\partial v} = x.\frac{\partial g}{\partial u} + \frac{\partial g}{\partial v}$

b) $\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y) + 7(x-y)f(x,y) = 0$ est équivalent à $(y-x).\left[\frac{\partial g}{\partial u} - 7g(u,v)\right] = 0$ soit $\frac{\partial g}{\partial u} - 7g(u,v) = 0$ ou $\frac{\partial g}{\partial u} = 7g(u,v)$

c) $\frac{\partial g}{\partial u} = 7g(u,v)$ a pour solutions $g(u,v) = K(v).\exp(7u)$ et les solutions de l'EDP sont donc les fonctions f telles que $f(x,y) = K(x+y).\exp(7xy)$

Exercice 5. *Recherche d'extremum, nature des points critiques*

Soit la fonction $f(x,y,z)$ de classe C^2 : $f(x,y,z) = (x-1)^2 + 3(y+1)^2 + 2(y+1)\,z + 3z^2$.

a) Dérivées partielles premières de f sur R^3 :

$\frac{\partial f}{\partial x} = 2(x-1)$, $\frac{\partial f}{\partial y} = 6\,(y+1) + 2z$, $\frac{\partial f}{\partial z} = 2(y+1) + 6z$.

Le (ou les) point(s) critique(s) de f sont caractérisés par $\frac{\partial f}{\partial x} = 2(x-1) = 0$, $\frac{\partial f}{\partial y} = 6(y+1) + 2z = 0$ et $\frac{\partial f}{\partial z} = 2(y+1) + 6z = 0$. Il y a donc un seul point critique (x=1,y=-1,z=0).

b) On calcule les dérivées partielles secondes de f :

$\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial y^2} = 6$, $\frac{\partial^2 f}{\partial z^2} = 6$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$, $\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 0$ et $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 2$. La matrice hessienne vaut donc $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix}$, les valeurs propres de cette matrice sont 2 (cf. $1^{ère}$ colonne), 4 et 8 (trace=14, déterminant=64) qui sont toutes positives. Il s'agit d'un minimum local.