Predictive control

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Outline

Some mathematical définitions

$$t = kT_e \Rightarrow x(t) \triangleq x(kT_e)$$

$$t - i = (k - i)T_e$$

Shift operator

$$q^{-i}x(t) = x(t - i)$$

$$\mathrm{P}(q^{-1}) = p_0 + p_1 q^{-1} + p_2 q^{-2} \ldots + p_n q^{-n_p}$$

$$\mathrm{P}(q^{-1})x(t) = p_0 x(t) + p_1 x(t-1) + p_2 x(t-2) + \cdots + p_n x(t-n)$$

Derivation

$$\frac{\partial P(q^{-1})x(t)}{\partial x(t-i)} = p_i$$

Class of systems

$$\begin{cases} A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t)+\mathrm{v(t)} & \textit{Model of the system} \\ \mathrm{D}(q^{-1})v(t) = C(q^{-1})\gamma(t) & \textit{Disturbance model} \end{cases}$$

Z transform

$$Y(z^{-1}) = \frac{z^{-d-1}B(z^{-1})}{A(z^{-1})}U(z^{-1}) + \frac{1}{A(z^{-1})}V(z^{-1})$$

$$\mathrm{V}(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})}$$

Disturbance Former Filter

Class of systems

$$A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2} + ..\ a_{n_a} q^{-n_a}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + b_2 q^{-2} + ..\ b_{n_b} q^{-n_b}$$

$$C(q^{-1}) = c_0 + c_1 q^{-1} + c_2 q^{-2} + ..\ c_{n_c} q^{-n_c}$$

$$D(q^{-1}) = 1 + d_1 q^{-1} + d_2 q^{-2} + ..\ d_{n_d} q^{-n_d}$$

Disturbance Former Filter

Step $D(q^{-1}) = 1 - q^{-1}$

Ramp $D(q^{-1}) = (1 - q^{-1})^2$

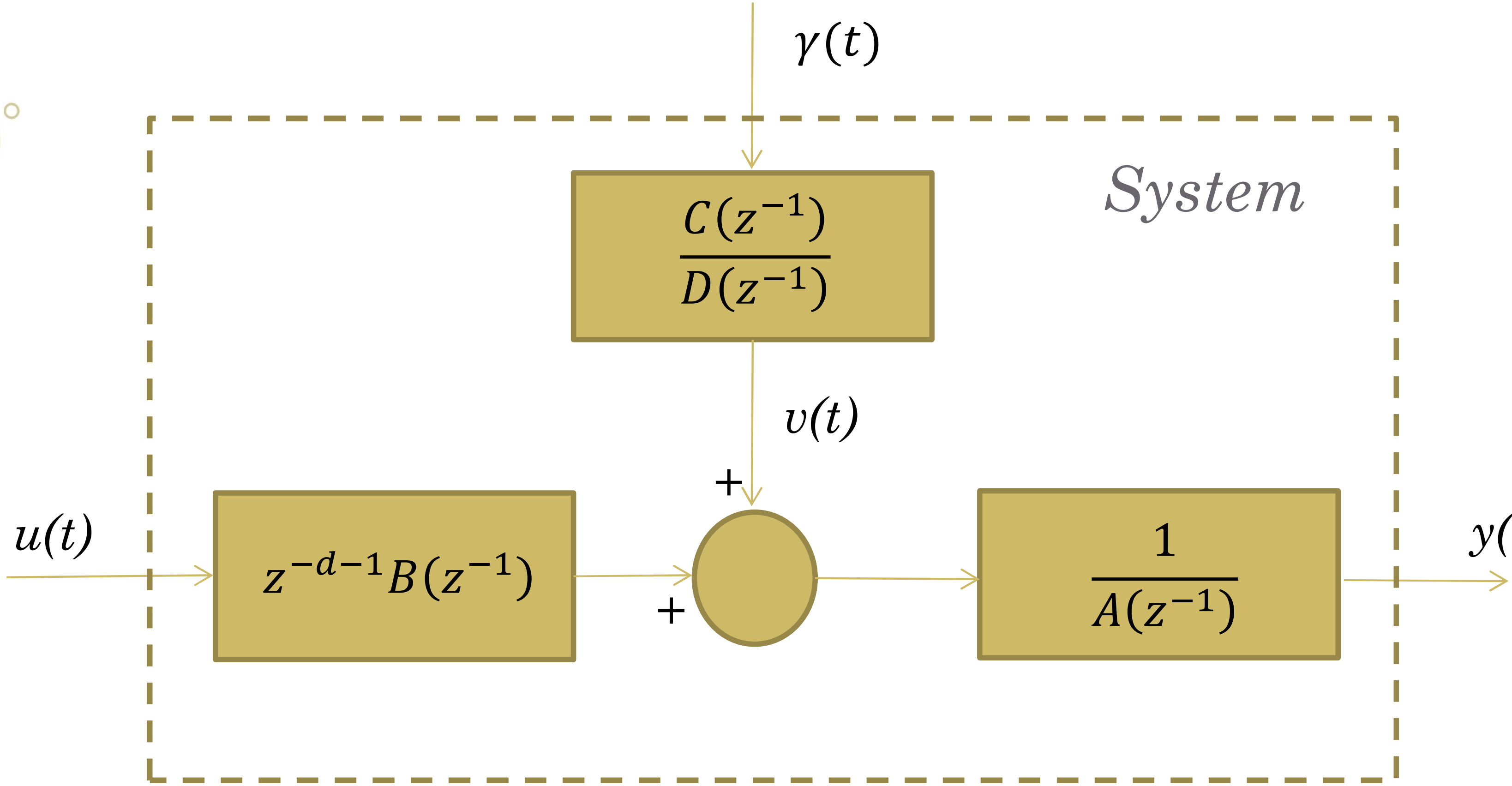
Sinus $D(q^{-1}) = 1 - 2\cos(wT_e)\, q^{-1} + q^{-2}$

!

*The **nature** of the disturbance is **completely** described by* $D(q^{-1})$

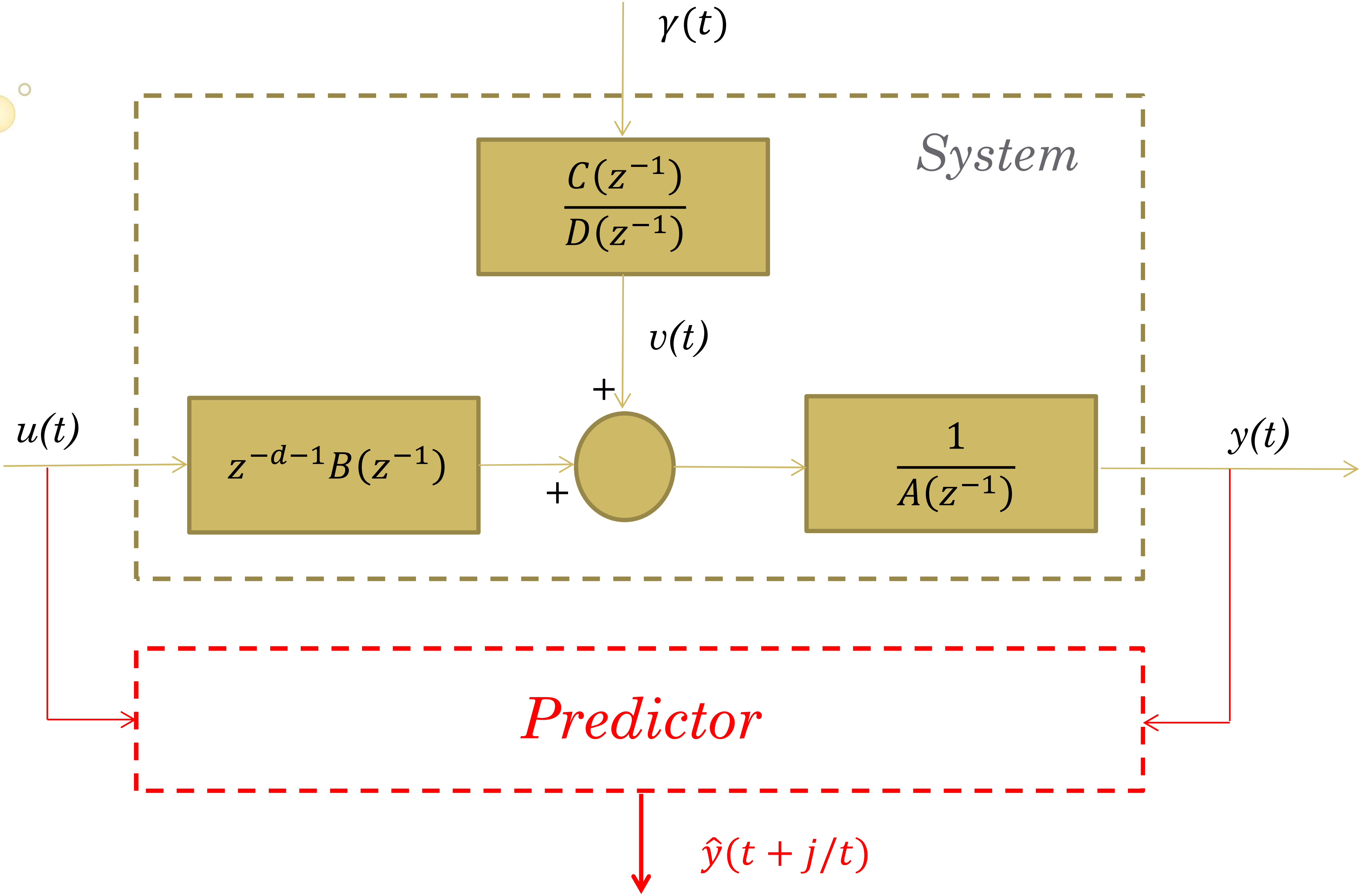
$C(q^{-1})$ *only acts as a filter*

Class of systems



$$Y(z^{-1}) = \frac{z^{-d-1}B(z^{-1})}{A(z^{-1})}U(z^{-1}) + \frac{C(z^{-1})}{A(z^{-1})D(z^{-1})}\gamma(z^{-1})$$

Linear prediction



Optimal Linear Prediction

$\hat{y}(t+j/t)$ *Optimal output prediction of* $y(t+j)$ *using the avalaible measurements at time t*

$\tilde{y}(t+j/t) = y(t+j) - \hat{y}(t+j/t)$ *Optimal output prediction error*

Properties of an optimal prediction

$\varepsilon\{\tilde{y}(t+j/t)\} = 0$ *No bias*

$\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\}$ *minimal*

Decomposition tools

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t) + \mathrm{v}(t)$$

We operate by $D(q^{-1})$ $\longrightarrow$

$$D(q^{-1})A(q^{-1})y(t) = q^{-d-1}D(q^{-1})B(q^{-1})u(t) + D(q^{-1})\,\mathrm{v}(t)$$

At $t = t+j$ $\longrightarrow$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}D(q^{-1})B(q^{-1})u(t+j) + D(q^{-1})\,\mathrm{v}(t+j)$$

Using disturbance model $\longrightarrow$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}D(q^{-1})B(q^{-1})u(t+j) + C(q^{-1})\,\gamma(t+j)$$

$$y(t+j) = \frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j) + \frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$$

Decomposition tools

$$y(t+j) = \frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j) + \frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$$

Depends on

The past values of u(t) : known

The future values of u(t) : can be known

Depends on

The past values of $\gamma(t)$: avalaible

The future values of $\gamma(t)$: unknown

Why $\gamma(t)$ is avalaible à time t ?

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t) + \mathrm{v}(t)$$

$$\downarrow$$

$$\mathrm{v}(t) = A(q^{-1})y(t) - q^{-d-1}B(q^{-1})u(t)$$

Decomposition tools

Why $\gamma(t)$ is avalaible à time t ?

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t) + \mathrm{v}(t)$$

$$\downarrow$$

$$\mathrm{v}(t) = A(q^{-1})y(t) - q^{-d-1}B(q^{-1})u(t)$$

$$\downarrow$$

$$D(q^{-1})\mathrm{v}(t) = D(q^{-1})A(q^{-1})y(t) - q^{-d-1}B(q^{-1})D(q^{-1})u(t)$$

$$\downarrow$$

$$\gamma(\mathrm{t}) = \frac{D(q^{-1})A(q^{-1})}{C(q^{-1})}y(t) - \frac{B(q^{-1})D(q^{-1})}{C(q^{-1})}u(t-d-1)$$

$\gamma(t)$ can be calculated using the values of y(t) and u(t)

Decomposition tools

$$y(t+j)=\frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j)+\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(\mathrm{t+j})$$

$$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(\mathrm{t+j}) \quad \textit{depends on} \quad \underbrace{\gamma(\mathrm{t+j}),\gamma(\mathrm{t+j-1}),\ \ldots,\ \gamma(\mathrm{t+1})}_{\textit{unpredictible}},\underbrace{\gamma(\mathrm{t}),\gamma(\mathrm{t-1}),\ldots}_{\textit{Can be calculated}}$$

Objective *Separate the unpredictable part and the avalaible part*

A polynomial division

$$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})} = E_j(q^{-1}) + q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}$$

$$E_j(q^{-1}) = e_0 + e_1 q^{-1} + e_2 q^{-2} + .. \, e_{n_{ej}} q^{-ne}$$

$$F_j(q^{-1}) = f_0 + f_1 q^{-1} + f_2 q^{-2} + .. \, f_{n_{fj}} q^{-nfj}$$

Another useful formulation

$$C(q^{-1}) = A(q^{-1})D(q^{-1})E_j(q^{-1}) + q^{-j}F_j(q^{-1})$$

Remark ***Simular to the diophantine equation (useful for Matlab / Simulink implementation)***

A polynomial division

At which rank to we decide to stop the division ? *nej = j -1*

$$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j) = E_j(q^{-1})\gamma(t+j) + q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$$

$$E_j(q^{-1})\gamma(t+j) = e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)$$ *Unpredictible part*

$$q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j) = \frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t)$$ *Avalaible part*

$$nej = \mathrm{j} - 1$$

$$nef \leq \max(n_a + n_d - 1, n_c + j)$$

Using the polynomial division

$$A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})u(t+j) + v(t+j)$$

$$\downarrow$$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})D(q^{-1})u(t+j) + D(q^{-1})\, v(t+j)$$

$$\downarrow$$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})D(q^{-1})u(t+j) + C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$E(q^{-1})D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j) + E(q^{-1})\, C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$\{C(q^{-1}) - q^{-j}F(q^{-1})\}y(t+j) = q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j) + E(q^{-1})\, C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$\begin{aligned} C(q^{-1})y(t+j) = {} & q^{-j}F(q^{-1})y(t+j) + q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j) \\ & + E(q^{-1})\, C(q^{-1})\gamma(t+j) \end{aligned}$$

Using the polynomial division

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1) + E(q^{-1})\,\gamma(t+j)$$

unpredictible

The optimal prediction

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

Is the prediction optimal ?

$$\tilde{y}(t+j/t) = y(t+j) - \hat{y}(t+j/t)$$

$$\tilde{y}(t+j/t) = E(q^{-1})\,\gamma(t+j) = e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)$$

Mean value

$$\varepsilon\{\tilde{y}(t+j/t)\} = \left\{e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)\right\}$$

$$= \varepsilon\left(\sum_{k=0}^{j-1}\{e_k\gamma(t+j-k)\}\right)$$

$$= \sum_{k=0}^{j-1}\{e_k\varepsilon(\gamma(t+j-k))\}$$

$$=0$$

Is the prediction optimal ?

$$
\begin{aligned}
\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\} &= \varepsilon\left\{\left(e_0\gamma(t+j)+e_1\gamma(t+j-1)+..+e_{j-1}\gamma(t+1)\right)^2\right\} \\
&= \varepsilon\left\{\sum_{i=0}^{j-1}\sum_{k=0}^{j-1} e_i e_k \gamma(t+j-i)\gamma(t+j-k)\right\} \\
&= \sum_{i=0}^{j-1} {e_i}^2\, \varepsilon\{(\gamma(t))^2\} \\
&= \sum_{i=0}^{j-1} {e_i}^2\, \sigma^2
\end{aligned}
$$

Conclusion

No bias

$$\varepsilon\{\tilde{y}(t+j/t)\} = 0$$

Minimal Variance

$$\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\} = \sum_{i=0}^{j-1} e_i^{\,2}\,\sigma^2$$

Prediction dynamics imposed by $C(q^{-1})$

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

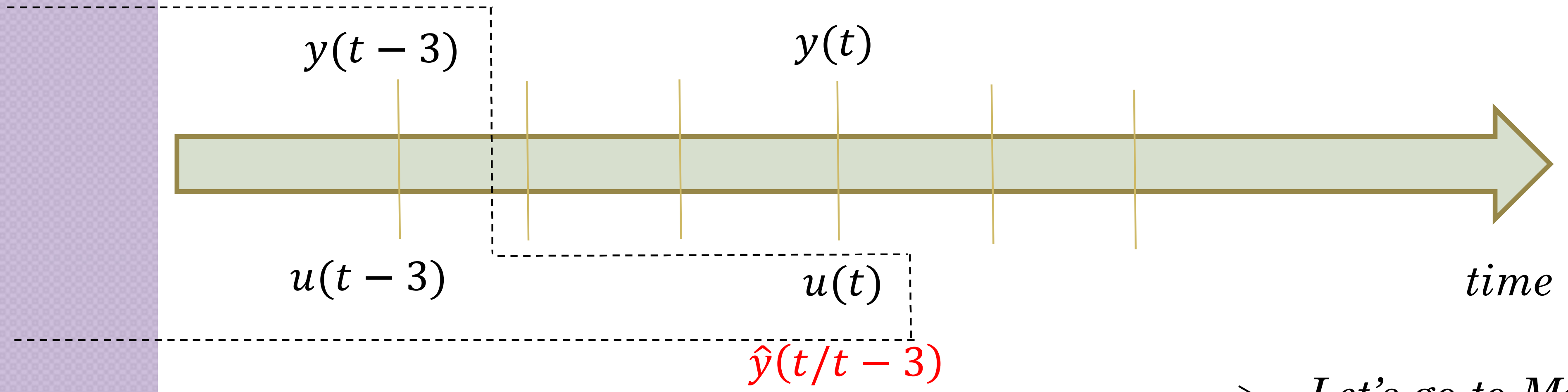
Exercice using Matlab / Simulink

The optimal prediction ***is not causal***

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

$\longrightarrow$ *Depends on the future values of the control variable*
Can not be simulated using Simulink

We are going to simulate $\hat{y}(t/t-j)$



$\longrightarrow$ *Let's go to Matlab*