$$\vec{B}(x,y,z) = \begin{pmatrix} \boxed{-xy} \\ xz \\ z(x^2+y^2) \end{pmatrix}$$

a) $$\iint_{S_+} \vec{B}\cdot\vec{n}\,dS = \iint_{S_+} \begin{pmatrix} -xy \\ xH \\ H(x^2+y^2) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS$$

$$= H \iint_{S_+} (x^2+y^2)\,dS = H \int_{r=0}^{R} \int_{\theta=0}^{2\pi} r^2 \times r\,dr\,d\theta$$

$$= H \int_0^R r^3\,dr \times \int_0^{2\pi} d\theta = H \left[\frac{r^4}{4}\right]_0^R \times 2\pi = \frac{\pi}{2} H R^4$$

$$\iint_{S_-} \vec{B}\cdot\vec{n}\,dS = \iint_{S_-} \begin{pmatrix} -xy \\ -xH \\ -H(x^2+y^2) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS$$

$$= H \iint_{S_-} (x^2+y^2)\,dS = \frac{\pi}{2} H R^4$$

$$\iint_{S_R} \vec{B}\cdot\vec{n}\,dS = \iint_{S_R} \begin{pmatrix} -R^2\cos(\theta)\sin(\theta) \\ R\cos(\theta)\,z \\ z\,R^2 \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} dS = \iint_{S_R} \left(-R^2\sin(\theta)\cos^2(\theta) + R\cos(\theta)\sin(\theta)\,z\right) R\,d\theta\,dz$$

$$= -R^3 \int_0^{2\pi} \sin\theta\cos^2\theta\,d\theta \int_{-H}^{H} dz + R^2 \int_0^{2\pi} \cos\theta\sin\theta\,d\theta \cancel{\int_{-H}^{H} z\,dz} = -R^3 \left[-\frac{\cos^3\theta}{3}\right]_0^{2\pi} \times 2H = 0$$

donc $$\oiint_{\partial V} \vec{B}\cdot\vec{dS} = 2 \times \frac{\pi}{2} H R^4 = \pi H R^4$$

b) $$\operatorname{div}\vec{B} = \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}\left(z(x^2+y^2)\right)$$

$$= -y + 0 + x^2 + y^2$$

$$\iiint_V \operatorname{div}\vec{B}\,dV = \iiint_V (x^2+y^2-y)\,dV = \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\int_{z=-H}^{H} (r^2 - r\sin\theta)\,r\,dr\,d\theta\,dz$$

$$= \int_0^R r^3\,dr \int_0^{2\pi} d\theta \int_{-H}^{H} dz + \int_0^R (-r^2)\,dr \cancel{\int_0^{2\pi} \sin\theta\,d\theta} \int_{-H}^{H} dz$$

$$= \left[\frac{r^4}{4}\right]_0^R \times 2\pi \times 2H = \pi H R^4$$

$$= \oiint_{\partial V} \vec{B}\cdot\vec{dS}$$

c) $$\oint_{\partial S_+} \vec{B}\cdot\vec{dl} = \int_0^{2\pi} \begin{pmatrix} -R^2\sin\theta\cos\theta \\ R\cos(\theta)\,H \\ HR^2 \end{pmatrix} \cdot \begin{pmatrix} -R\sin(\theta) \\ R\cos(\theta) \\ 0 \end{pmatrix} d\theta = \int_0^{2\pi} R^2\left(R\sin^2\theta\cos\theta + H\cos^2(\theta)\right) d\theta$$

$$= R^3 \int_0^{2\pi} \cos\theta\sin^2\theta\,d\theta + HR^2 \int_0^{2\pi} \cos^2\theta\,d\theta = R^3 \cancel{\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi}} + HR^2 \int_0^{2\pi} \frac{\cos 2\theta + 1}{2}\,d\theta$$

$$= HR^2 \left[\frac{\sin 2\theta}{4} + \frac{1}{2}\right]_0^{2\pi} = \pi H R^2$$

d) $$\vec{rot}\,\vec{B} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} -xy \\ xz \\ z(x^2+y^2) \end{pmatrix} = \begin{pmatrix} 2yz - x \\ -2xz \\ z + x \end{pmatrix}$$

$$\iint_{S_+} \vec{rot}\,\vec{B}\cdot\vec{dS} = \iint_{S_+} \begin{pmatrix} 2yz - x \\ -2xz \\ z + x \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS = \iint_{S_+} (z+x)\,dS = \iint_{S_+} (H+x)\,dS$$

$$= H \iint_{S_+} dS + \underbrace{\iint_{S_+} x\,dS}_{=0} = \pi H R^2 = \oint_{\partial S_+} \vec{B}\cdot\vec{dl}$$