Predictive control

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Outline

Some mathematical définitions

$t = kT_e$ ⇒ $x(t) \triangleq x(kT_e)$

$t - i = (k - i)T_e$

Shift operator

$q^{-i}x(t) = x(t-i)$

$\mathrm{P}(q^{-1}) = p_0 + p_1 q^{-1} + p_2 q^{-2} \ldots + p_n q^{-n_p}$

$\mathrm{P}(q^{-1})x(t) = p_0 x(t) + p_1 x(t-1) + p_2 x(t-2) + \cdots + p_n x(t-n)$

Derivation

$\frac{\partial P(q^{-1})x(t)}{\partial x(t-i)} = p_i$

Class of systems

$$\begin{cases} A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t) + \mathrm{v}(t) \\ \mathrm{D}(q^{-1})v(t) = C(q^{-1})\gamma(t) \end{cases}$$

Model of the system

Disturbance model

⇓ *Z transform*

$$Y(z^{-1}) = \frac{z^{-d-1}B(z^{-1})}{A(z^{-1})}U(z^{-1}) + \frac{1}{A(z^{-1})}V(z^{-1})$$

$$\mathrm{V}(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})}$$ ⇒ *Disturbance Former Filter*

Class of systems

$$A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-1} + .. \, a_{n_a} q^{-1}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + b_2 q^{-1} + .. \, b_{n_b} q^{-1}$$

$$C(q^{-1}) = c_0 + c_1 q^{-1} + c_2 q^{-1} + .. \, c_{n_c} q^{-1}$$

$$D(q^{-1}) = 1 + d_1 q^{-1} + d_2 q^{-1} + .. \, d_{n_d} q^{-1}$$

Disturbance Former Filter

Step $D(q^{-1}) = 1 - q^{-1}$

Ramp $D(q^{-1}) = (1 - q^{-1})^2$

Sinus $D(q^{-1}) = 1 - 2\cos(wT_e)\, q^{-1} + q^{-2}$

!

*The **nature** of the disturbance is **completely** described by* $D(q^{-1})$

$C(q^{-1})$ *only acts as a filter*

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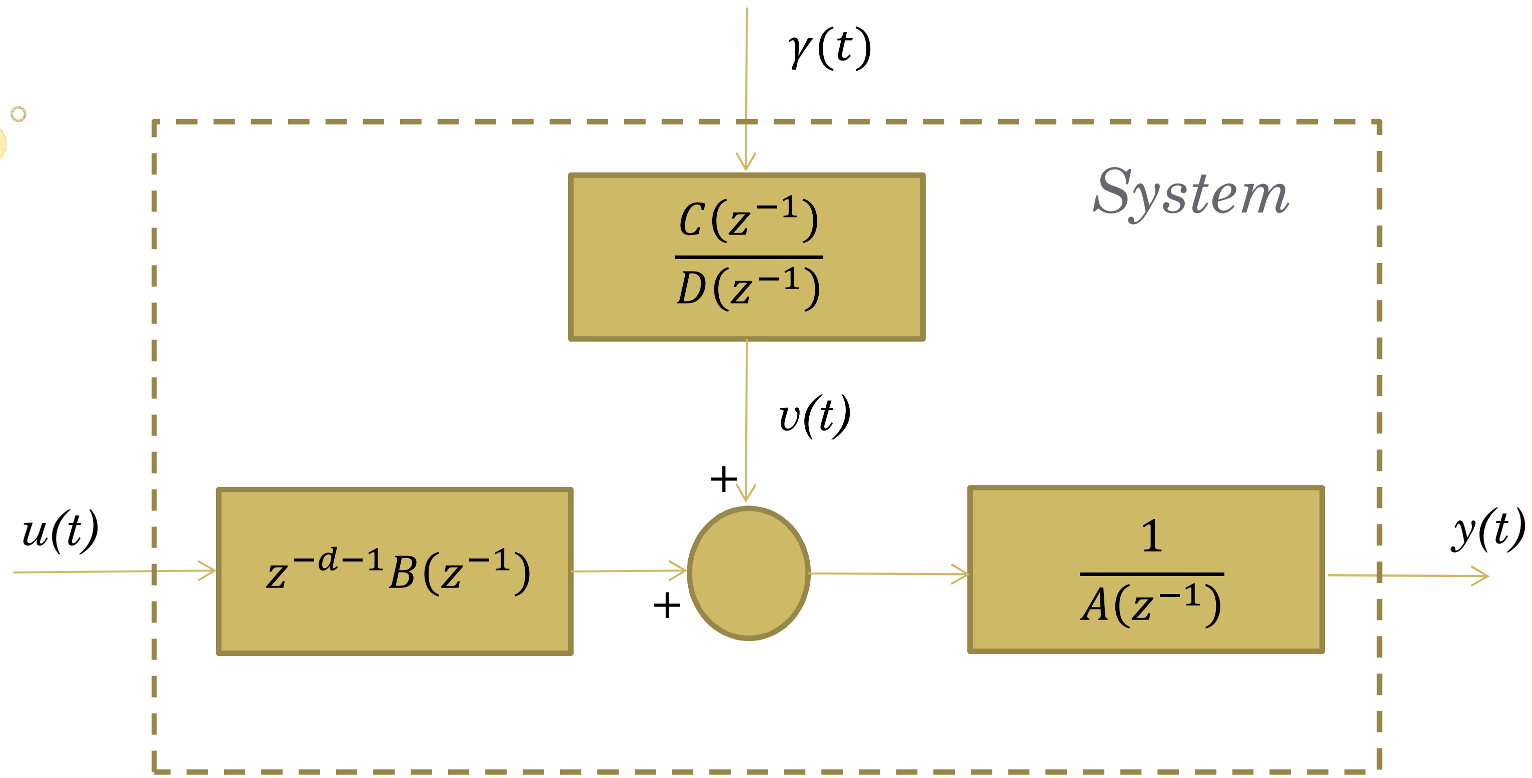
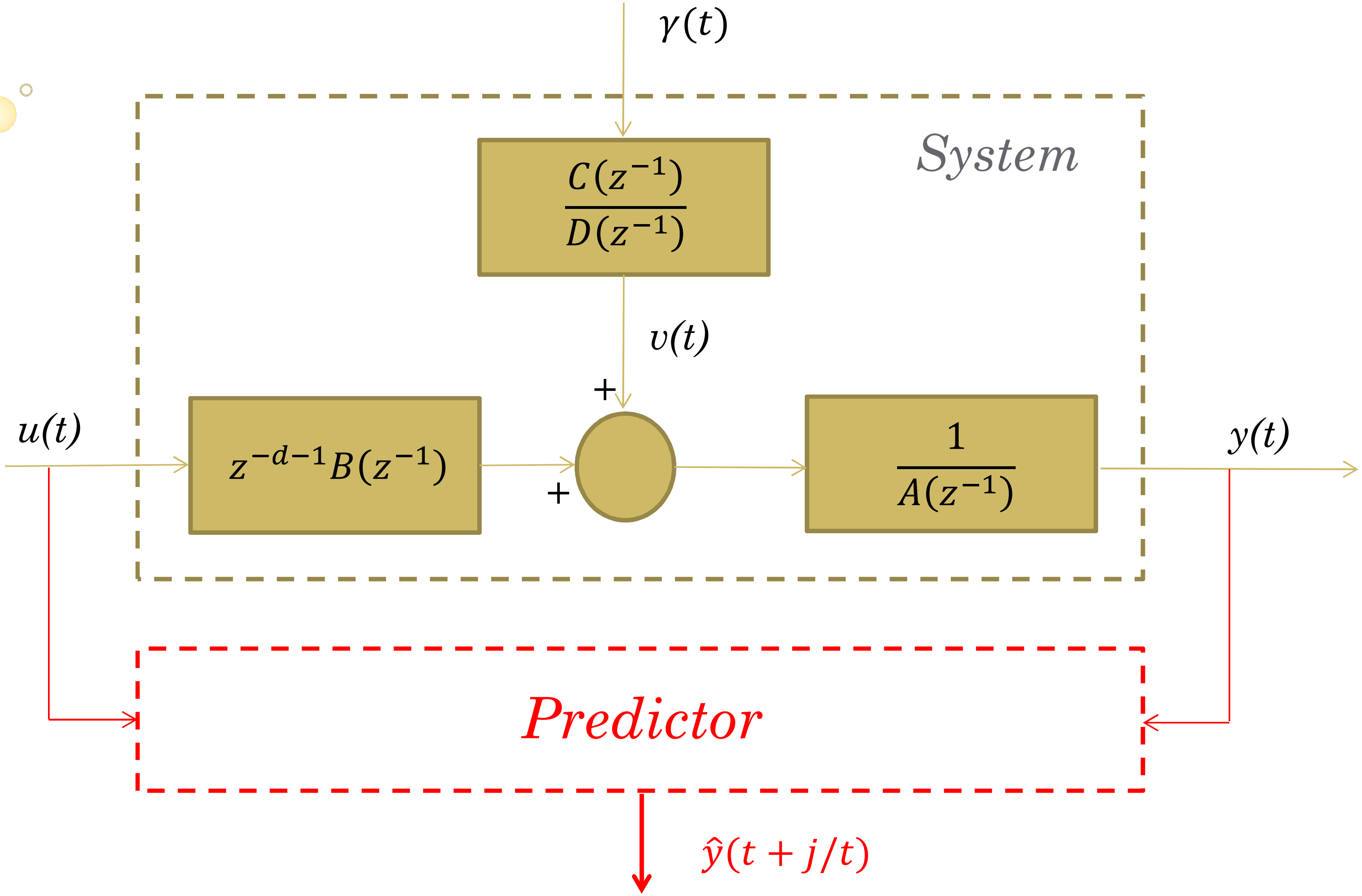
Class of systems



$$Y(z^{-1}) = \frac{z^{-d-1}B(z^{-1})}{A(z^{-1})}U(z^{-1}) + \frac{C(z^{-1})}{A(z^{-1})D(z^{-1})}\gamma(z^{-1})$$

Linear prediction



Optimal Linear Prediction

$\hat{y}(t+j/t)$ *Optimal output prediction of* $y(t+j)$ *using the avalaible measurements at time t*

$\tilde{y}(t+j/t) = y(t+j) - \hat{y}(t+j/t)$ *Optimal output prediction error*

Properties of an optimal prediction

$\varepsilon\{\tilde{y}(t+j/t)\} = 0$ *No bias*

$\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\}$ *minimal*

Decomposition tools

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t) + \text{v}(t)$$

We operate → by $\text{D}(q^{-1})$

$$D(q^{-1})A(q^{-1})y(t) = q^{-d-1}D(q^{-1})B(q^{-1})u(t) + D(q^{-1})\,\text{v}(t)$$

At t = t+j →

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}D(q^{-1})B(q^{-1})u(t+j) + D(q^{-1})\,\text{v}(t+j)$$

Using disturbance → model

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}D(q^{-1})B(q^{-1})u(t+j) + C(q^{-1})\,\gamma(\text{t+j})$$

$$y(t+j) = \frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j) + \frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(\text{t+j})$$

Decomposition tools

$$y(t+j) = \frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j) + \frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$$

Depends on

The past values of u(t) : known

The future values of u(t) : can be known

Depends on

The past values of $\gamma(t)$ *: avalaible*

The future values of $\gamma(t)$*: unknown*

Why $\gamma(t)$ *is avalaible à time t ?*

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t) + \mathrm{v}(t)$$

$$\downarrow$$

$$\mathrm{v}(t) = A(q^{-1})y(t) - q^{-d-1}B(q^{-1})u(t)$$

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Decomposition tools

Why $\gamma(t)$ is avalaible à time t ?

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t) + \mathrm{v}(t)$$

$$\downarrow$$

$$\mathrm{v}(t) = A(q^{-1})y(t) - q^{-d-1}B(q^{-1})u(t)$$

$$\downarrow$$

$$D(q^{-1})\mathrm{v}(t) = D(q^{-1})A(q^{-1})y(t) - q^{-d-1}B(q^{-1})D(q^{-1})u(t)$$

$$\downarrow$$

$$\gamma(t) = \frac{D(q^{-1})A(q^{-1})}{C(q^{-1})}y(t) - \frac{B(q^{-1})D(q^{-1})}{C(q^{-1})}u(t-d-1)$$

$\gamma(t)$ can be calculated using the values of $y(t)$ and $u(t)$

Decomposition tools

$$y(t+j) = \frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j) + \frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$$

$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$ *depends on* $\gamma(t+j), \gamma(t+j-1), \ldots, \gamma(t+1), \gamma(t), \gamma(t-1), \ldots$

- $\gamma(t+j), \gamma(t+j-1), \ldots, \gamma(t+1)$: *unpredictible*
- $\gamma(t), \gamma(t-1), \ldots$: *Can be calculated*

Objective *Separate the unpredictable part and the avalaible part*

A polynomial division

$$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})} = E_j(q^{-1}) + q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}$$

$$E_j(q^{-1}) = e_0 + e_1 q^{-1} + e_2 q^{-2} + ..\ e_{n_{ej}} q^{-nej}$$

$$F_j(q^{-1}) = f_0 + f_1 q^{-1} + f_2 q^{-2} + ..\ f_{n_{fj}} q^{-nfj}$$

Another useful formulation

$$C(q^{-1}) = A(q^{-1})D(q^{-1})E_j(q^{-1}) + q^{-j}F_j(q^{-1})$$

Remark ***Simular to the diophantine equation (useful for Matlab / Simulink implementation)***

A polynomial division

At which rank to we decide to stop the division ? nej = j -1

$$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j) = E_j(q^{-1})\gamma(t+j) + q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$$

$$E_j(q^{-1})\gamma(t+j) = e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)$$ Unpredictible part

$$q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j) = \frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t)$$ Avalaible part

$$nej = j - 1$$

$$nef \leq \max(n_a + n_d - 1, n_c + j)$$

Using the polynomial division

$$A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})u(t+j)+v(t+j)$$

$$\downarrow$$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})D(q^{-1})u(t+j)+ D(q^{-1})\ v(t+j)$$

$$\downarrow$$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})D(q^{-1})u(t+j)+ C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$E(q^{-1})D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j)+ E(q^{-1})\ C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$\{C(q^{-1}) - q^{-j}F(q^{-1})\}y(t+j) = q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j)+ E(q^{-1})\ C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$C(q^{-1})y(t+j) = q^{-j}F(q^{-1})y(t+j) + q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j) + E(q^{-1})\ C(q^{-1})\gamma(t+j)$$

Using the polynomial division

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1) + E(q^{-1})\,\gamma(t+j)$$

↓

unpredictible

The optimal prediction

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

Is the prediction optimal ?

$$\tilde{y}(t+j/t) = y(t+j) - \hat{y}(t+j/t)$$

$$\tilde{y}(t+j/t) = E(q^{-1})\,\gamma(t+j) = e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)$$

Mean value

$$\varepsilon\{\tilde{y}(t+j/t)\} = \{e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)\}$$

$$= \varepsilon\left(\sum_{k=0}^{j-1}\{e_k\gamma(t+j-k)\}\right)$$

$$= \sum_{k=0}^{j-1}\{e_k\varepsilon(\gamma(t+j-k))\}$$

$$=0$$

Is the prediction optimal ?

$$\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\} = \varepsilon\left\{\left(e_0\gamma(t+j) + e_1\gamma(t+j-1)+..+e_{j-1}\gamma(t+1)\right)^2\right\}$$

$$= \varepsilon\left\{\sum_{i=0}^{j-1}\sum_{k=0}^{j-1} e_i e_k \gamma(t+j-i)\gamma(t+j-k)\right\}$$

$$= \sum_{i=0}^{j-1} e_i^{\,2}\, \varepsilon\{(\gamma(t))^2\}$$

$$= \sum_{i=0}^{j-1} e_i^{\,2}\, \sigma^2$$

Conclusion

No bias

$$\varepsilon\{\tilde{y}(t+j/t)\} = 0$$

Minimal Variance

$$\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\} = \sum_{i=0}^{j-1} {e_i}^2 \, \sigma^2$$

Prediction dynamics imposed by $C(q^{-1})$

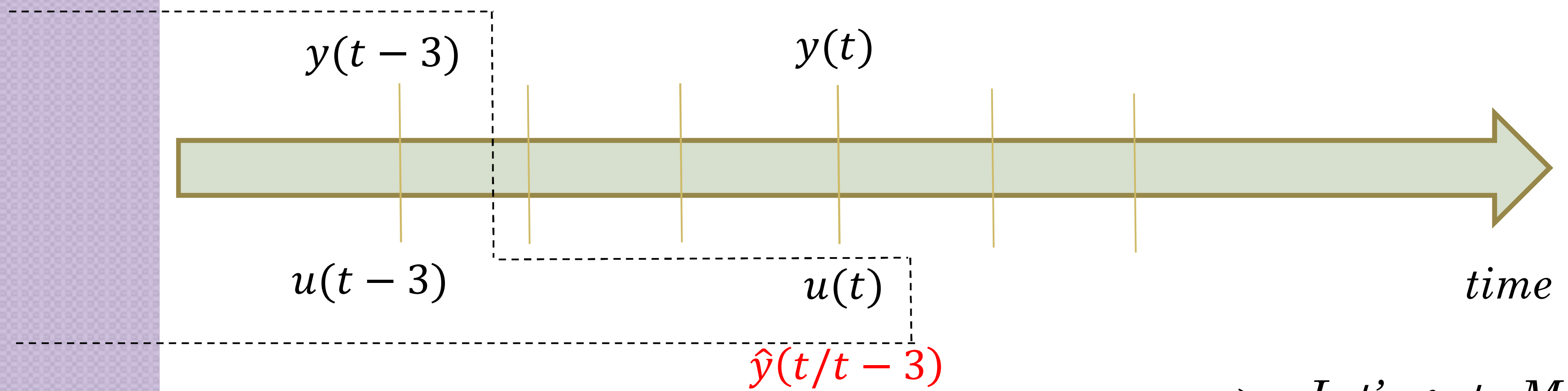
$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})} y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})} u(t+j-d-1)$$

The optimal prediction ***is not causal***

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

→ *Depends on the future values of the control variable*
Can not be simulated using Simulink

We are going to simulate $\hat{y}(t/t-j)$



→ *Let's go to Matlab*