



# Predictive control

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# *Schedule*

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*1 – Optimal prediction for control design purposes*

*2 – Criteria and derivation of the criteria*

*3 – Linear Time Invariant controller*

*4 – Input / Output performances*



*Optimal prediction for control design  
purposes*

## New parametrization of the predictor

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1) + E(q^{-1})\gamma(t+j)$$

Let's have a look at  $\frac{B(q^{-1})E(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1)$

It contains terms  $\{u(t+j-d-1) \dots u(t)\}$  and  $\{u(t-1) \ u(t-2) \dots\}$

↑  
*Future control values  
that have to be calculated*

↑  
*Already available  
at time  $t$*

## New parametrization of the predictor

*We are going to separate the future and the past values of the control values*

*We can do that with a **second polynomial division***

$$\frac{B(q^{-1})E(q^{-1})}{C(q^{-1})} = G_{j-d}(q^{-1}) + q^{-j+d} \frac{H_{j-d}(q^{-1})}{C(q^{-1})}$$

$$G(q^{-1}) = g_0 + g_1 q^{-1} + \dots + g_{j-d-1} q^{-j+d+1}$$

$$H(q^{-1}) = h_0 + h_1 q^{-1} + \dots + h_{nh} q^{-nh}$$



*The degree  $(j-d-1)$  of  $G(q^{-1})$  plays an important role*

## New parametrization of the predictor

Hence

$$\begin{aligned}
 & \frac{B(q^{-1})E(q^{-1})}{C(q^{-1})} D(q^{-1})u(t+j-d-1) \\
 &= G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + q^{-j+d} \frac{H_{j-d}(q^{-1})}{C(q^{-1})} D(q^{-1})u(t+j-d-1) \\
 &= (g_0 + g_1q^{-1} + \dots + g_{j-d-1}q^{-j+d+1})D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})} D(q^{-1})u(t-1) \\
 &= \underbrace{\{g_0u(t+j-d-1) + g_1u(t+j-d) + \dots + g_{j-d-1}u(t)\}}_{\text{Future control values}} + \underbrace{\frac{H_{j-d}(q^{-1})}{C(q^{-1})} D(q^{-1})u(t-1)}_{\text{Last control values}}
 \end{aligned}$$

*Future control values  
(to be calculated)*

*Last control values  
(already known)*

## *New parametrization of the predictor*

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1) + E(q^{-1})\gamma(t+j)$$



$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1) + E(q^{-1})\gamma(t+j)$$



## To summarize

$$G_{j-d}(q^{-1})D(q^{-1})u(t + j - d - 1)$$

*Only depends on  
the actual and future control values*

$$\frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t - 1)$$

*Is completely known at time  $t$*

$$E(q^{-1})\gamma(t + j)$$

*Is unpredictable*



## The optimal $j$ -step predictor

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1) + E(q^{-1})v(t+j)$$



*What is available for prediction ?*

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)$$



$$\hat{y}(t+j/t) = G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \hat{y}_0(t+j/t)$$

with 
$$\hat{y}_0(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)$$

## A set of predictors

$$\hat{y}(t + j/t) = G_{j-d}(q^{-1})D(q^{-1})u(t + j - d - 1) + \hat{y}_0(t + j/t)$$

From  $j = d+1$  to  $j = h_p$

$$\hat{y}(t + d + 1/t) = g_0 D(q^{-1})u(t) + \hat{y}_0(t + d + 1/t)$$

$$\hat{y}(t + d + 2/t) = g_0 D(q^{-1})u(t + 1) + g_1 D(q^{-1})u(t) + \hat{y}_0(t + d + 2/t)$$

$$\hat{y}(t + h_p/t) = g_0 D(q^{-1})u(t + h_p - d - 1) + \dots + g_{h_p-d-1} D(q^{-1})u(t) + \hat{y}_0(t + d + 2/t)$$

## Matricial Form of the set of predictors

$$\begin{pmatrix} \hat{y}(t+d+1/t) \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+d+1/t) \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$



$$\hat{Y}(t+h_p/t) = \Phi D(q^{-1})U(t+h_p-d-1) + \hat{Y}_0(t+h_p/t)$$



*Criteria and derivation of the criteria*



## The criteria

Find the control vector  $U(t + h_c - 1)$  that minimizes

$$J(U(t + h_c - 1)) = \varepsilon \left( \sum_{j=h_i}^{h_p} \{y(t + j) - y^*(t + j)\}^2 \right) + \varepsilon \left( \sum_{j=h_i}^{h_p} \lambda \{D(q^{-1})u(t + j - h_i)\}^2 \right)$$

with  $U(t + h_c - d - 1) = [u(t) \quad \dots \quad u(t + h_c - 1)]$

$$u(t + j) = 0 \quad \forall j \geq h_c$$

## An equivalent criteria

The control vector  $U(t + h_c - 1)$  that minimizes

$$J(U(t + h_c - 1)) = \varepsilon \left( \sum_{j=h_i}^{h_p} \{y(t + j) - y^*(t + j)\}^2 \right) + \varepsilon \left( \sum_{j=h_i}^{h_p} \lambda \{D(q^{-1})u(t + j - h_i)\}^2 \right)$$

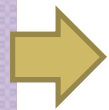
Also minimizes

$$\hat{J}(U(t + h_c - 1)) = \sum_{j=h_i}^{h_p} \{\hat{y}(t + j/t) - y^*(t + j)\}^2 + \sum_{j=h_i}^{h_p} \lambda \{D(q^{-1})u(t + j - h_i)\}^2$$

## New matricial expression for the set of predictors

$$\begin{pmatrix} \hat{y}(t+d+1/t) \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+d+1/t) \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

1 We need the predictors from  $h_i$  to  $h_p$



We suppress the first lines

$$\begin{pmatrix} \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} \overline{g_0} & \cdots & \overline{0} \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$



## New matricial expression for the set of predictors

2

Control values are nul if  $j \geq h_c$



We suppress the last columns of  $\Phi$

$$\begin{pmatrix} \hat{y}(t+d+1/t) \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+d+1/t) \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

## New matricial expression for the set of predictors

*Final expression*

$$\begin{pmatrix} \hat{y}(t + h_i/t) \\ \hat{y}(t + h_i + 1/t) \\ \vdots \\ \hat{y}(t + h_p/t) \end{pmatrix} = \begin{pmatrix} g_{h_i-d-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_{h_p-d-h_c} \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_c-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t + h_i/t) \\ \hat{y}_0(t + h_i + 1/t) \\ \vdots \\ \hat{y}_0(t + h_p/t) \end{pmatrix}$$

$$\hat{Y}(t + h_p/t) = \Phi_r D(q^{-1}) U(t + h_c - 1) + \hat{Y}_0(t + h_p/t)$$

## The criteria

$$\hat{J}(U(t + h_c - 1)) = \sum_{j=h_i}^{h_p} \{\hat{y}(t + j/t) - y^*(t + j)\}^2 + \sum_{j=h_i}^{h_p} \lambda \{D(q^{-1})u(t + j - h_i)\}^2$$
$$= \|\hat{Y}(t + h_p/t) - \hat{Y}^*(t + h_p/t)\|^2 + \lambda \|D(q^{-1})U(t + h_c - 1)\|^2$$

with

$$\hat{Y}^*(t + h_p/t) = \begin{pmatrix} \hat{y}^*(t + h_i/t) \\ \hat{y}^*(t + h_i + 1/t) \\ \vdots \\ \hat{y}^*(t + h_p/t) \end{pmatrix}$$

## The criteria

$$\hat{J}(U(t + h_c - 1)) = \|\hat{Y}(t + h_p/t) - \hat{Y}^*(t + h_p/t)\|^2 + \lambda \|D(q^{-1})U(t + h_c - 1)\|^2$$

$$\frac{\partial \hat{J}(U(t + h_c - d - 1))}{\partial (U(t + h_c - 1))} = \frac{\partial \hat{J}(U(t + h_c - d - 1))}{\partial (D(q^{-1})U(t + h_c - 1))} \frac{\partial (D(q^{-1})U(t + h_c - 1))}{\partial (U(t + h_c - 1))}$$

$$= \frac{\partial \hat{J}(U(t + h_c - d - 1))}{\partial (D(q^{-1})U(t + h_c - 1))}$$

## The criteria

$$\begin{aligned}\hat{J}(U(t+h_c-1)) &= \|\hat{Y}(t+h_p/t) - \hat{Y}^*(t+h_p/t)\|^2 + \lambda \|D(q^{-1})U(t+h_c-1)\|^2 \\ &= \|\Phi_r D(q^{-1})U(t+h_c-1) + \hat{Y}_0(t+h_p/t) - \hat{Y}^*(t+h_p/t)\|^2 + \lambda \|D(q^{-1})U(t+h_c-1)\|^2\end{aligned}$$

$$\begin{aligned}\frac{\partial \hat{J}(U(t+h_c-1))}{\partial (D(q^{-1})U(t+h_c-1))} &= 2\Phi_r^T (\Phi_r D(q^{-1})U(t+h_c-1) + \hat{Y}_0(t+h_p/t) - \hat{Y}^*(t+h_p/t)) \\ &\quad + 2\lambda D(q^{-1})U(t+h_c-1)\end{aligned}$$

## The criteria

$$\frac{\partial \hat{J}(U(t + h_c - d - 1))}{\partial (D(q^{-1})U(t + h_c - 1))} = 0$$



$$(\Phi_r^T \Phi_r + \lambda I) D(q^{-1})U(t + h_c - 1) = \Phi_r^T (Y^*(t + h_p) - \hat{Y}_0(t + h_p/t))$$



$$D(q^{-1})U(t + h_c - 1) = (\Phi_r^T \Phi_r + \lambda I)^{-1} \Phi_r^T (Y^*(t + h_p) - \hat{Y}_0(t + h_p/t))$$

## Receding horizon concept

We only keep the first element of  $D(q^{-1})U(t + h_c - 1)$  e.g  $D(q^{-1})u(t)$

Why ?

If we keep and apply  $D(q^{-1})U(t + h_c - 1)$  the system operates in *open loop* during sampling periods



$$D(q^{-1})U(t + h_c - 1) = (\Phi_r^T \Phi_r + \lambda I)^{-1} \Phi_r^T (Y^*(t + h_p) - \hat{Y}_0(t + h_p/t))$$



$$D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j (y^*(t + j) - \hat{y}_0(t + j/t))$$

$\gamma_j$  Elements of the first line of  $(\Phi_r^T \Phi_r + \lambda I)^{-1} \Phi_r^T$





# *Linear Time Invariant Controller*

# Linear Time Invariant Equivalent Controller

$$D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j (y^*(t+j) - \hat{y}_0(t+j/t))$$

Introduce the expression of  $\hat{y}_0(t+j/t)$

We operate by  $C(q^{-1})$


$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j C(q^{-1})(y^*(t+j) - \hat{y}_0(t+j/t))$$

$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_j C(q^{-1})\hat{y}_0(t+j/t)$$

# Linear Time Invariant Equivalent Controller

$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_j C(q^{-1})\hat{y}_0(t+j/t)$$

$$C(q^{-1})\hat{y}_0(t+j/t) = F(q^{-1})y(t) + H_{j-d}(q^{-1})D(q^{-1})u(t-1)$$


$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_j F(q^{-1})y(t)$$

$$- \sum_{j=h_i}^{h_p} \gamma_j H_{j-d}(q^{-1})D(q^{-1})u(t-1)$$

# Linear Time Invariant Equivalent Controller

$$C(q^{-1})D(q^{-1})u(t) + \sum_{j=h_i}^{h_p} \gamma_j H_{j-d}(q^{-1})D(q^{-1})u(t-1) + \sum_{j=h_i}^{h_p} \gamma_j F(q^{-1})y(t) = \sum_{j=h_i}^{h_p} \gamma_j C(q^{-1})y^*(t+j)$$

$$R(q^{-1})y(t) + S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1)$$



$$R(q^{-1}) = \sum_{j=h_i}^{h_p} \gamma_j F(q^{-1})$$

$$S(q^{-1}) = D(q^{-1}) \left\{ C(q^{-1}) + \sum_{j=h_i}^{h_p} \gamma_j q^{-1} H_{j-d}(q^{-1}) \right\}$$

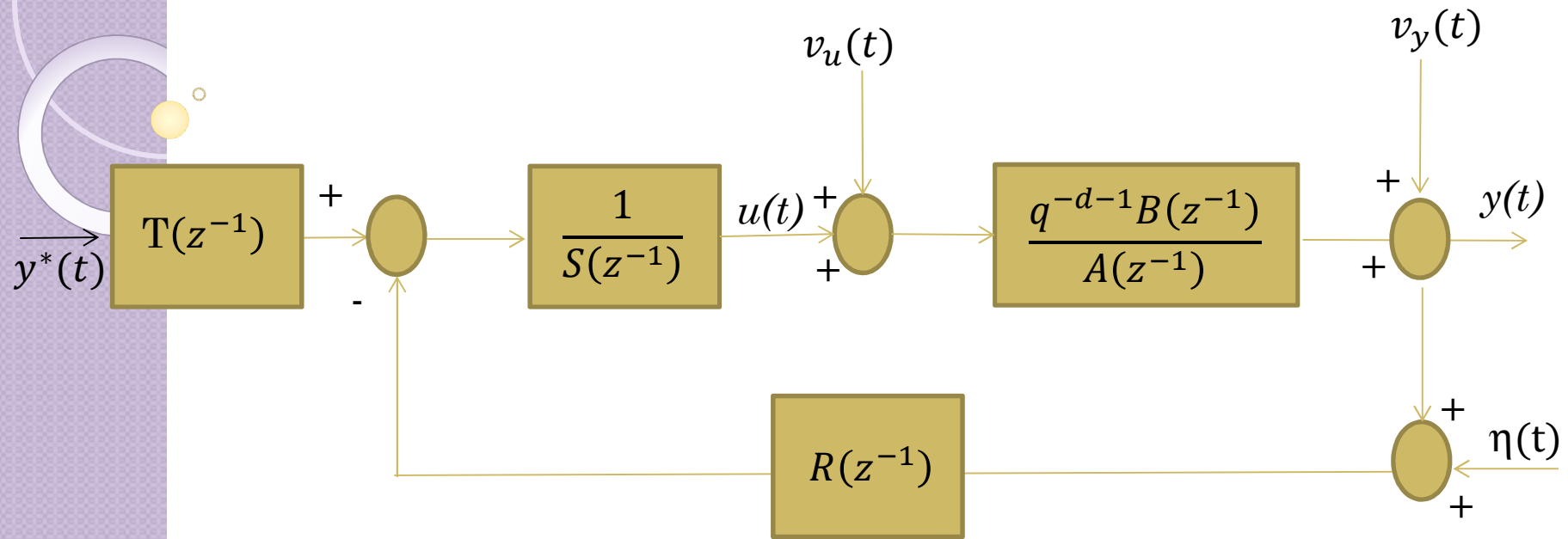
$$R(q^{-1}) = C(q^{-1}) \sum_{j=h_i}^{h_p} \gamma_j$$



*Input - Output performances*



## Closed-loop performances



$v_u(t)$  *Input disturbance (low frequency)*

$v_y(t)$  *Output disturbance (low frequency)*

$y^*(t)$  *Reference sequence*

$\eta(t)$  *Noise measurements (high frequency)*

## Closed-loop performances

### Output performances

$$y(t) = \frac{q^{-d-1}B(q^{-1})T(q^{-1})}{P_c(z^{-1})}y^*(t) \quad \text{Tracking dynamics}$$

$$+ \frac{A(q^{-1})S(q^{-1})}{P_c(z^{-1})}v_y(t)$$

$$+ \frac{B(q^{-1})S(q^{-1})}{P_c(z^{-1})}v_u(t)$$

$$+ \frac{S(q^{-1})}{P_c(z^{-1})}\eta(t)$$

Disturbance rejection dynamics



## Closed-loop performances

### Input performances

$$u(t) = \frac{A(q^{-1})T(q^{-1})}{P_c(z^{-1})} y^*(t)$$

$$+ \frac{A(q^{-1})S(q^{-1})}{P_c(z^{-1})} v_y(t)$$

$$+ \frac{B(q^{-1})S(q^{-1})}{P_c(z^{-1})} v_u(t)$$

$$+ \frac{A(q^{-1})R(q^{-1})}{P_c(z^{-1})} \eta(t)$$

*Tracking dynamics*

*Disturbance rejection dynamics*

## *Design parameters*

$\lambda$	Input weighting
$h_i$	Initialization Horizon
$h_c$	Control Horizon
$h_p$	Prediction Horizon
$C(q^{-1})$	Prediction dynamics
$D(q^{-1})$	Disturbance Model

## Key properties

$$P_c(q^{-1}) = C(q^{-1})P_f(q^{-1})$$

Separation theorem

**Property 1**  $h_p = h_i = d + 1, h_c = 1, \lambda = 0$

$$\rightarrow P_f(q^{-1}) = \frac{1}{b_0} P_f(q^{-1})$$

**Minimal Variance Control**  
**(Lecture n°2)**

**Property 2**  $h_p = h_i = d + 1, h_c = 1, \lambda \neq 0$

**One step Ahead**  
**Predictive Control**  
**(Lecture n°2)**

## Key properties

**Property 3**  $h_i = n_b + d + 1, h_c = n_a + n_d, h_p > h_i + h_c, \lambda = 0$

$$\rightarrow P_f(q^{-1}) = 1$$

$$\rightarrow P_c(q^{-1}) = C(q^{-1})$$

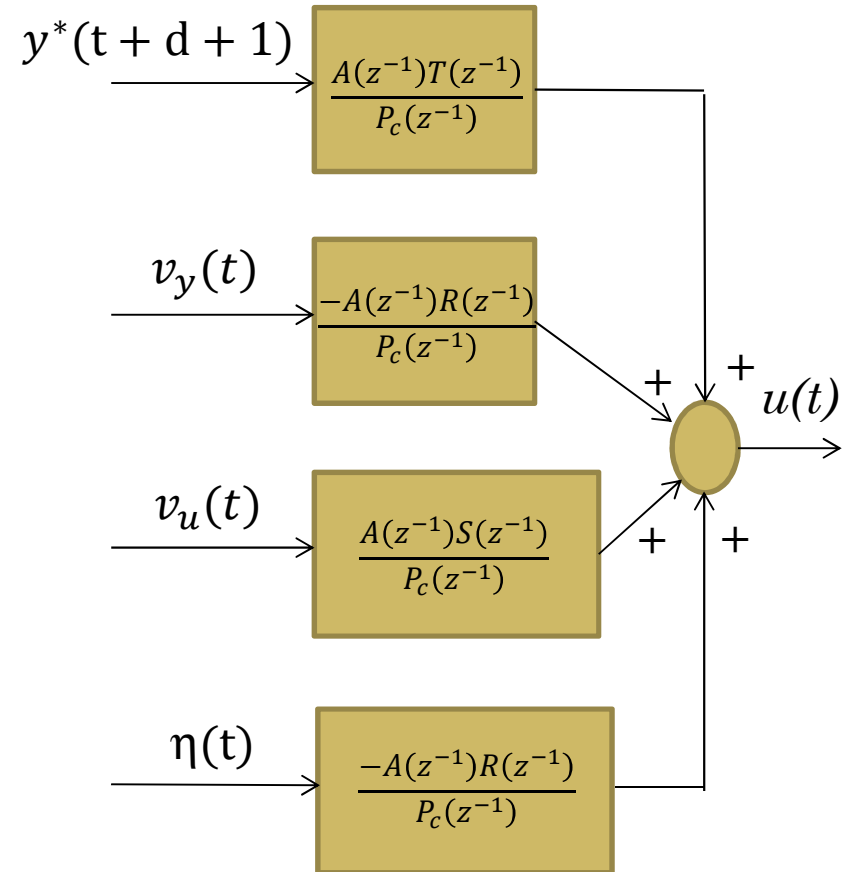
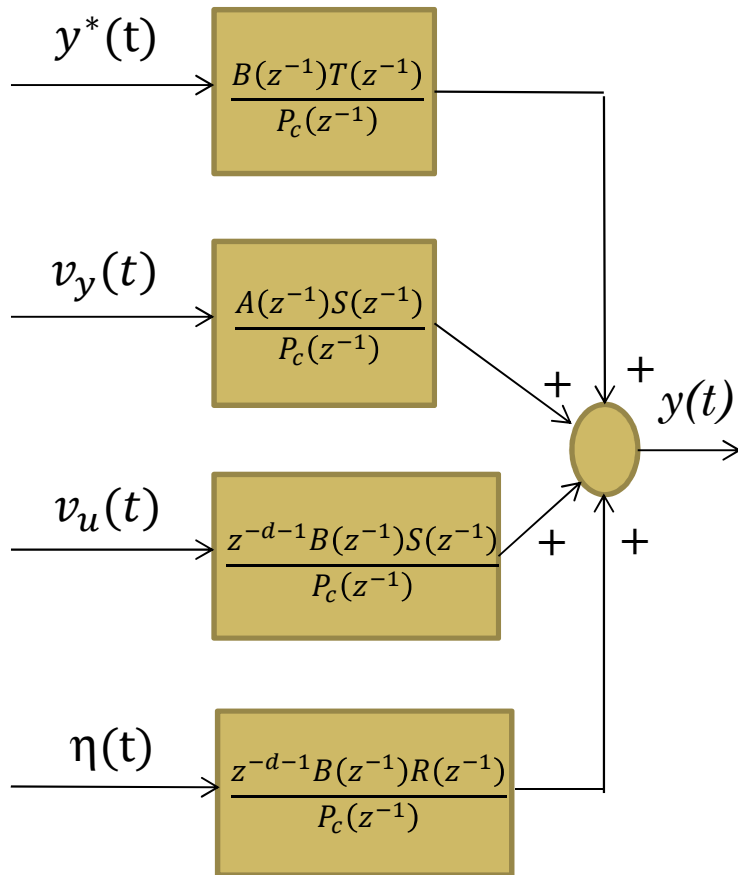
**Pole placement**

**Property 4**  $h_i = d + 1, h_c = 1, \lambda = 0, h_p \rightarrow \infty, D(q^{-1}) = 1 - q^{-1}$

$$\rightarrow P_f(q^{-1}) = A(q^{-1})$$

**Internal Model Control**

# Usual Sensitivity functions



## Usual Sensitivity functions

*Sensitivity function*

$$\Psi(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{P_c(z^{-1})} = \frac{y(z^{-1})}{v_y(z^{-1})} = \frac{u(z^{-1})}{v_u(z^{-1})}$$

*Complementary Sensitivity function*

$$\Gamma(z^{-1}) = \frac{B(z^{-1})R(z^{-1})}{P_c(z^{-1})} = \frac{y(z^{-1})}{\eta(z^{-1})}$$

*Sensitivity function  $\times$  Controller*

$$\Psi(z^{-1}) \frac{R(z^{-1})}{S(z^{-1})} = \frac{u(z^{-1})}{\eta(z^{-1})}$$

*Sensitivity function  $\times$  System*

$$\Psi(z^{-1}) \frac{B(z^{-1})}{A(z^{-1})} = \frac{y(z^{-1})}{v_u(z^{-1})}$$



# Controller Design

*Parameters choice* → *Shaping of the usual sensitivity functions*

## **Sensitivity function**

*High pass filter* → *Disturbance rejection  $v_y(t)$*   
*Bandwidth* → *Dynamics*  
*Modulus Margin* → *Robustness / modeling errors*

## **Complementary Sensitivity function**

*Low pass filter* → *Noise rejection /  $y(t)$*   
*Bandwidth* → *Dynamics*

## **Sensitivity function × Controller**

*Low pass filter* → *Noise rejection /  $u(t)$*   
*Bandwidth* → *Dynamics*

## **Sensitivity function × System**

*High pass filter* → *Disturbance rejection  $v_u(t)$*   
*Bandwidth* → *Dynamics*