Cours de Commande prédictive

ENSICAEN

2ème année Spécialité GPSE, majeure SATE

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Commande Prédictive

(Model Predictive Control – MPC)

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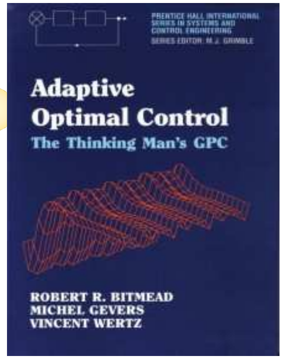
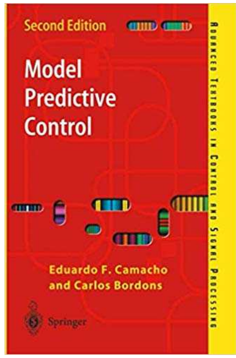
Plan

Références





Webinar Matlab

https://fr.mathworks.com/videos/series/understanding-model-predictive-control.html

principe – exemple concret - contraintes

D. Clarke, C. Mohtadi, P.S. Tuffs, Generalized predictive control – Part I, The basic algorithm, Automatica, 23 (2), 1987, pp 137 – 148

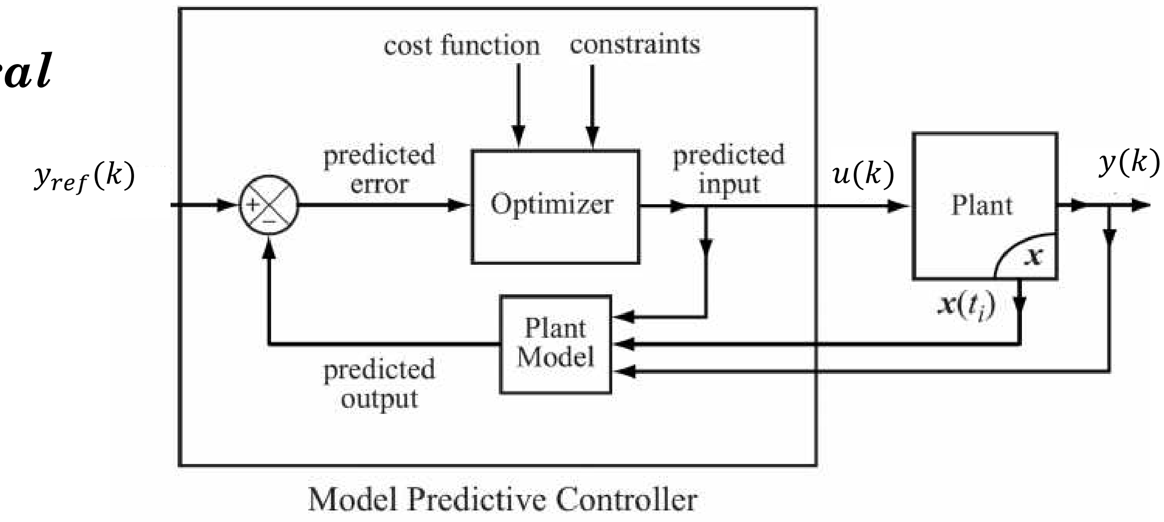
D. Clarke, C. Mohtadi, P.S. Tuffs, Generalized predictive control – Part II, Extensions and interpretations Automatica, 23 (2), 1987, pp 149 – 160

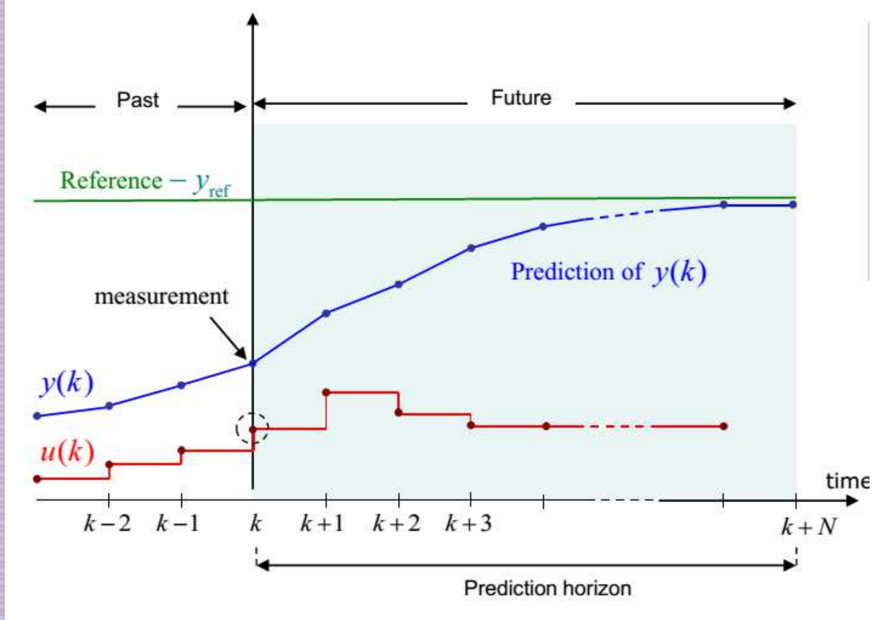
P. Dorléans, O. Gehan, E. Pigeon, M. M'Saad, M. Hertz and M. Desalle, Diameter regulation of an optical fiber using a generalized predictive control approach, in Proc. of 14 th World IFAC Congress, Pékin, 1998.

O. Gehan, J. Reuter, E. Pigeon and M. Pouliquen, Multivariable MPC algorithm with separated prediction horizons : application to simultaneous control of tension and drawing speed in optical fiber manufactoring processes, ECC 2018, Limassol, Cyprus, 2018.

Model Predictive Control

Principe Général





Plant Model

Permet de prédire les sorties futures

Optimizer

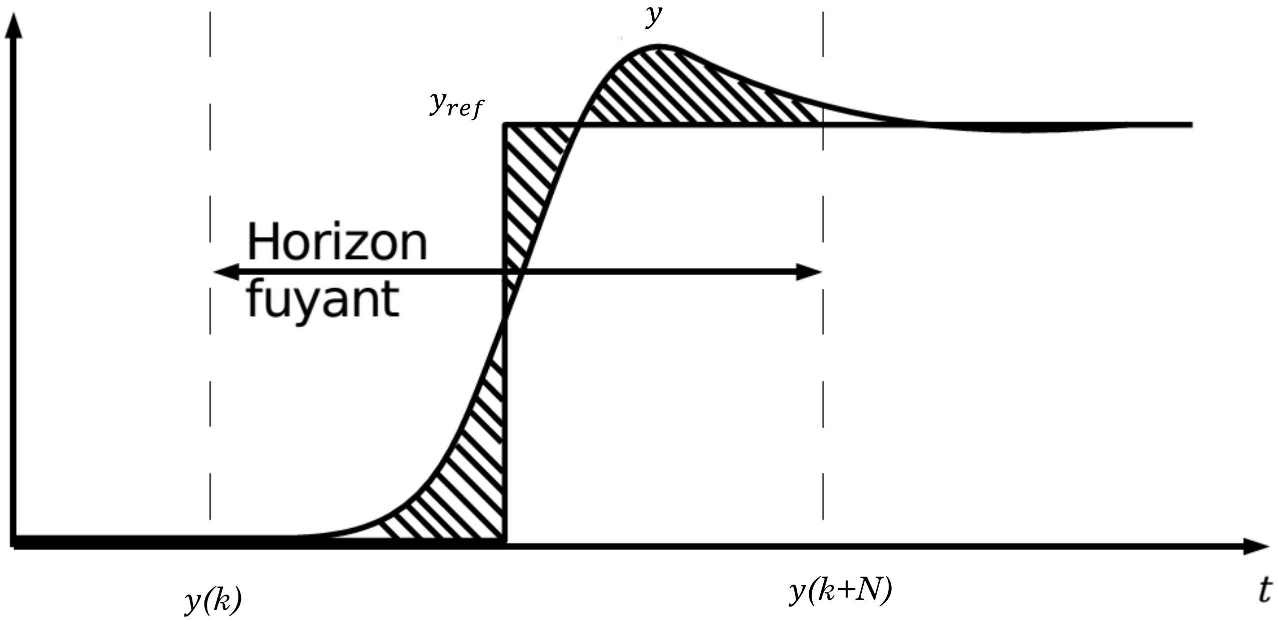
Détermine à chaque instant k les commandes futures à partir des sorties prédites et des contraintes

Minimisation d'une fonction « coût »

Horizon Fuyant

A t = k, seule la première commande u(k) est appliquée et l'optimisation est faite à nouveau au coup suivant

Principe Général de l'optimisation



Minimiser la surface hachurée ainsi que l'énergie de la commande u(t) pour y parvenir

Avantages / (PID ou autres lois de commande)

- *systèmes SISO ou MIMO   voir les 2 Applications*

- *prise en compte possible de contraintes lors de l'optimisation (bornes, gradients)*
  sur la commande
  sur l'état
  sur les sorties

- *gestion de la sensibilité de la commande au bruit de mesure*

- *possibilité de traiter des perturbations de natures différentes des échelons (seul cas traité par le PID) -→ ex : perturbation harmonique*

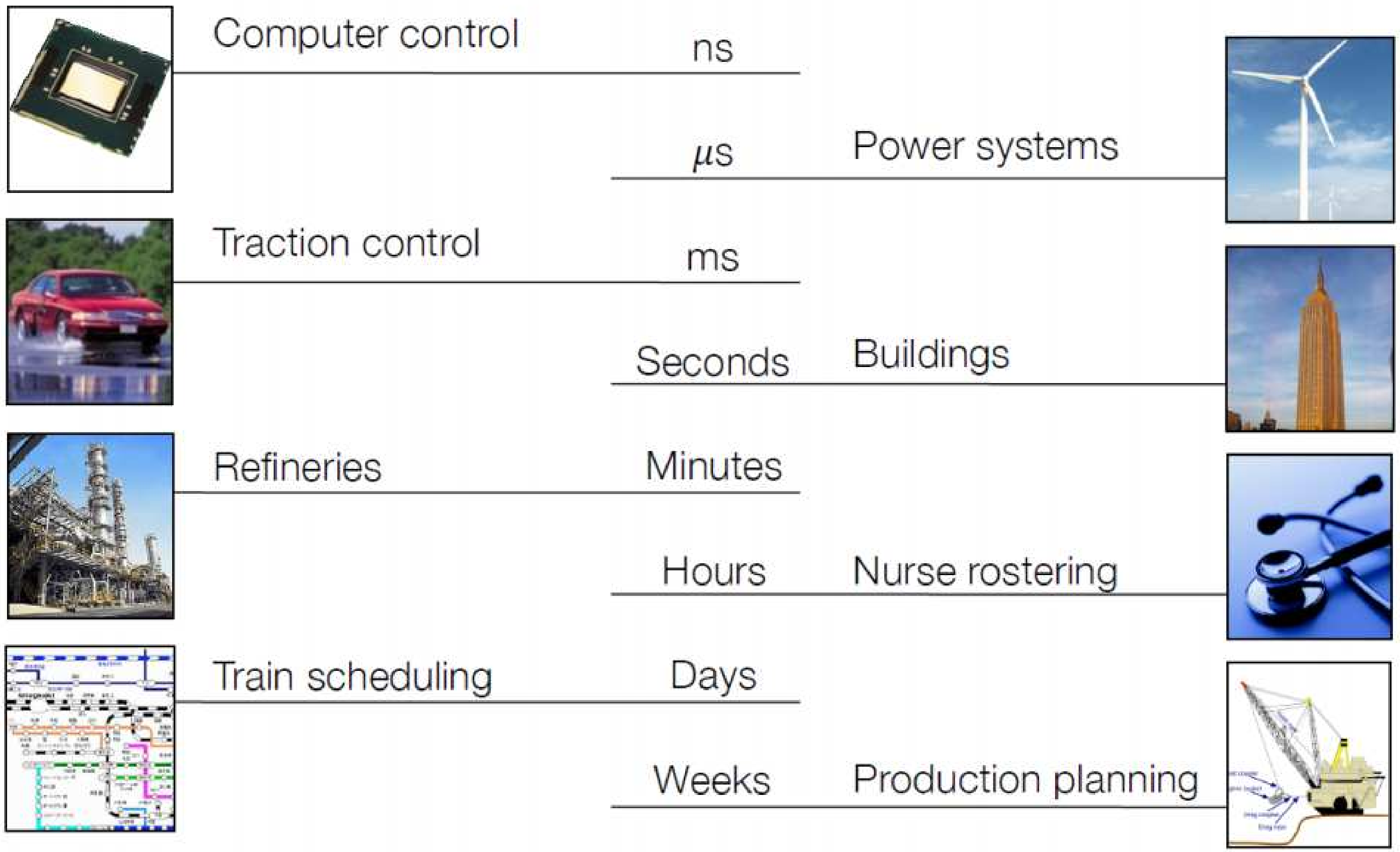
Caractéristiques

- *basé sur les modèles E/S (fonction de transfert) ou le modèle d'Etat*

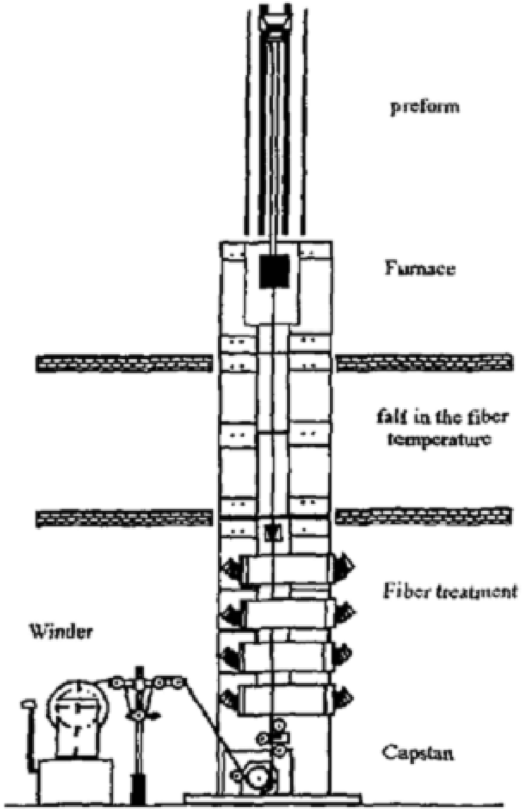
- *linéaire (Commande Prédictive à un pas, GPC) ou non – linéaire (N-MPC)*

- *<u>dans le cas LTI sans contraintes</u>, pas besoin d'optimiser à chaque pas, **on peut trouver le régulateur LTI hors ligne***

Quelques applications et ordre de grandeurs temporelles



Application SISO et MIMO (2 X 2)



Boucle SISO

Entrée : *vitesse d'enroulement (capstan)*

Sortie : *diamètre de la fibre (précision µm)*

Boucle MIMO

Entrées : *vitesse de descente préforme*
puissance de chauffe du four

Sorties : *vitesse de fibrage*
tension de la fibre

Predictive control

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Outline

I Introduction

II Linear optimal prediction

III One step ahead predictive control

Some mathematical définitions

$$t = kT_e \quad \Rightarrow \quad x(t) \triangleq x(kT_e)$$

$$t - i = (k - i)T_e$$

Shift operator

$$q^{-i}x(t) = x(t-i)$$

$$\mathrm{P}(q^{-1}) = p_0 + p_1 q^{-1} + p_2 q^{-2} \ldots + p_n q^{-n_p}$$

$$\mathrm{P}(q^{-1})x(t) = p_0 x(t) + p_1 x(t-1) + p_2 x(t-2) + \cdots + p_n x(t-n)$$

Derivation

$$\frac{\partial P(q^{-1})x(t)}{\partial x(t-i)} = p_i$$

Class of systems

$$\begin{cases} A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t)+\mathrm{v(t)} & \textit{Model of the system} \\ \mathrm{D}(q^{-1})v(t) = C(q^{-1})\gamma(t) & \textit{Disturbance model} \end{cases}$$

⇓ *Z transform*

$$Y(z^{-1}) = \frac{z^{-d-1}B(z^{-1})}{A(z^{-1})}U(z^{-1}) + \frac{1}{A(z^{-1})}V(z^{-1})$$

$\mathrm{V}(z^{-1}) = \frac{C(z^{-1})}{D(z^{-1})}$ ⇒ *Disturbance Former Filter*

Class of systems

$$A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2} + ..\ a_{n_a} q^{-n_a}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + b_2 q^{-2} + ..\ b_{n_b} q^{-n_b}$$

$$C(q^{-1}) = c_0 + c_1 q^{-1} + c_2 q^{-2} + ..\ c_{n_c} q^{-n_c}$$

$$D(q^{-1}) = 1 + d_1 q^{-1} + d_2 q^{-2} + ..\ d_{n_d} q^{-n_d}$$

Disturbance Former Filter

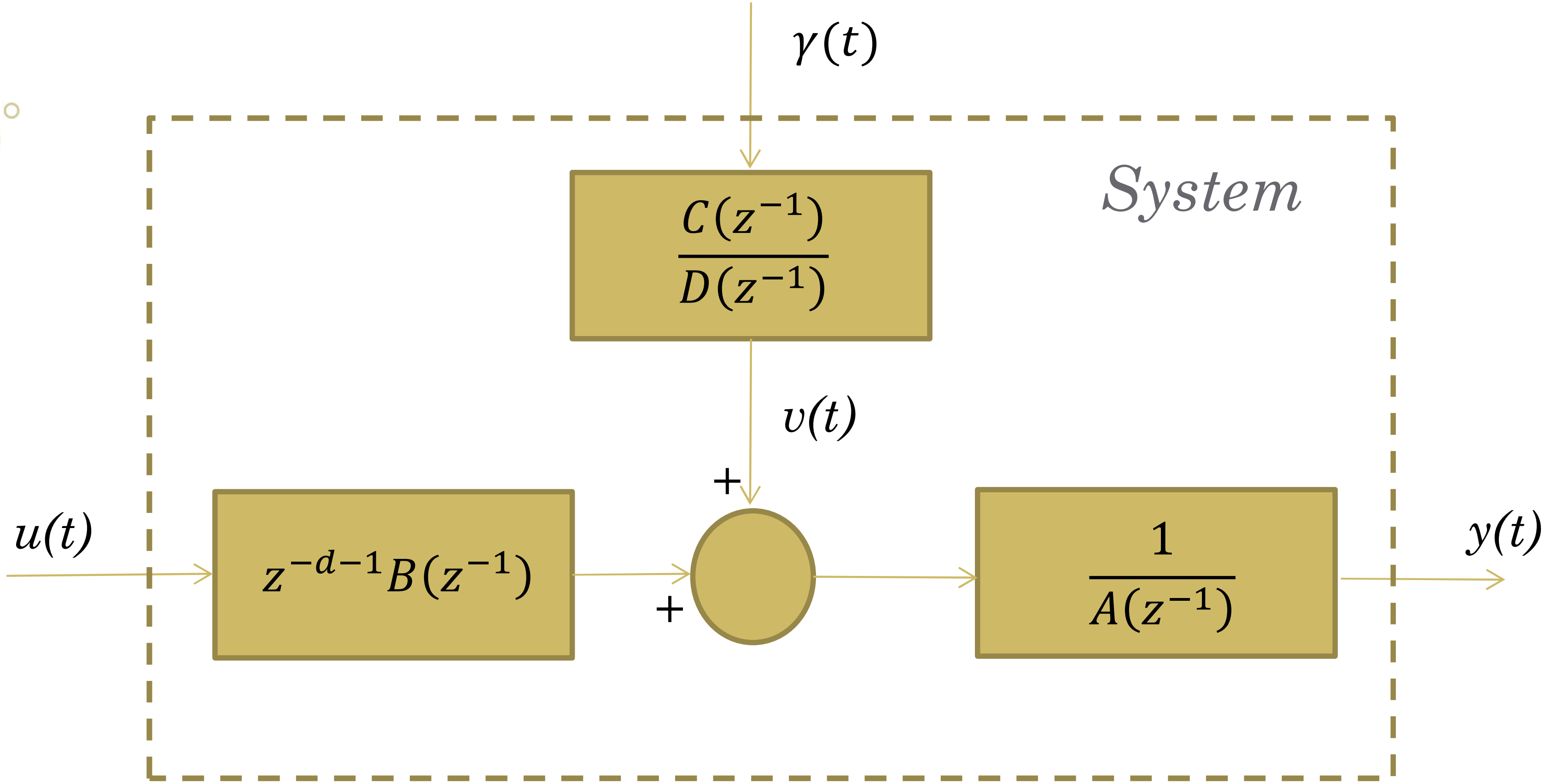
Step	$D(q^{-1}) = 1 - q^{-1}$
Ramp	$D(q^{-1}) = (1 - q^{-1})^2$
Sinus	$D(q^{-1}) = 1 - 2\cos(wT_e)\, q^{-1} + q^{-2}$

!

*The **nature** of the disturbance is **completely** described by* $D(q^{-1})$

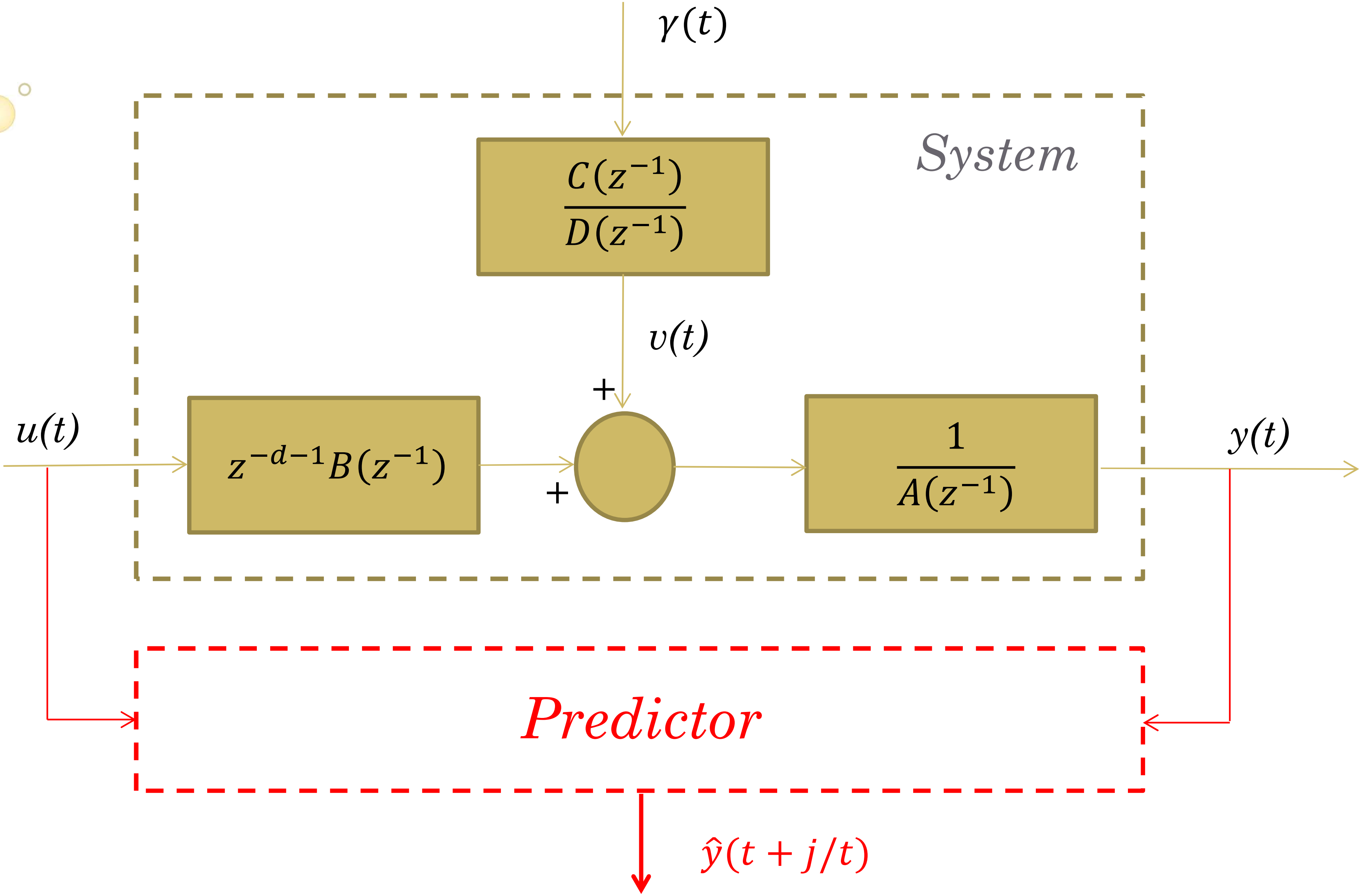
$C(q^{-1})$ *only acts as a filter*

Class of systems



$$Y(z^{-1}) = \frac{z^{-d-1}B(z^{-1})}{A(z^{-1})}U(z^{-1}) + \frac{C(z^{-1})}{A(z^{-1})D(z^{-1})}\gamma(z^{-1})$$

Linear prediction



Optimal Linear Prediction

$\hat{y}(t+j/t)$ *Optimal output prediction of* $y(t+j)$ *using the avalaible measurements at time t*

$\tilde{y}(t+j/t) = y(t+j) - \hat{y}(t+j/t)$ *Optimal output prediction error*

Properties of an optimal prediction

$\varepsilon\{\tilde{y}(t+j/t)\} = 0$ *No bias*

$\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\}$ *minimal*

Decomposition tools

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t)+\mathrm{v(t)}$$

We operate $\longrightarrow$ by $\mathrm{D}(q^{-1})$

$$D(q^{-1})A(q^{-1})y(t) = q^{-d-1}D(q^{-1})B(q^{-1})u(t)+ D(q^{-1})\ \mathrm{v(t)}$$

At t = t+j $\longrightarrow$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}D(q^{-1})B(q^{-1})u(t+j)+ D(q^{-1})\ \mathrm{v(t+j)}$$

Using disturbance $\longrightarrow$ model

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}D(q^{-1})B(q^{-1})u(t+j)+ C(q^{-1})\ \gamma\mathrm{(t+j)}$$

$$y(t+j) = \frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j)+\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma\mathrm{(t+j)}$$

Decomposition tools

$$y(t+j) = \frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j) + \frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(\mathrm{t+j})$$

Depends on

The past values of u(t) : known

The future values of u(t) : can be known

Depends on

The past values of $\gamma(t)$ *: avalaible*

The future values of $\gamma(t)$*: unknown*

Why $\gamma(t)$ *is avalaible à time t ?*

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t) + \mathrm{v(t)}$$

$$\downarrow$$

$$\mathrm{v(t)} = A(q^{-1})y(t) - q^{-d-1}B(q^{-1})u(t)$$

Decomposition tools

Why $\gamma(t)$ is avalaibe à time t ?

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})u(t)+\mathrm{v}(t)$$

$$\downarrow$$

$$\mathrm{v}(t) = A(q^{-1})y(t) - q^{-d-1}B(q^{-1})u(t)$$

$$\downarrow$$

$$D(q^{-1})\mathrm{v}(t) = D(q^{-1})A(q^{-1})y(t) - q^{-d-1}B(q^{-1})D(q^{-1})u(t)$$

$$\downarrow$$

$$\gamma(t) = \frac{D(q^{-1})A(q^{-1})}{C(q^{-1})}y(t) - \frac{B(q^{-1})D(q^{-1})}{C(q^{-1})}u(t-d-1)$$

$\gamma(t)$ can be calculated using the values of y(t) and u(t)

Decomposition tools

$$y(t+j) = \frac{q^{-d-1}B(q^{-1})}{A(q^{-1})D(q^{-1})}u(t+j) + \frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$$

$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$ *depends on*

$$\underbrace{\gamma(t+j),\gamma(t+j-1),\ \ldots,\ \gamma(t+1)}_{\text{unpredictible}},\underbrace{\gamma(t),\gamma(t-1),..}_{\text{Can be calculated}}$$

Objective *Separate the unpredictable part and the avalaible part*

A polynomial division

$$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})} = E_j(q^{-1}) + q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}$$

$$E_j(q^{-1}) = e_0 + e_1 q^{-1} + e_2 q^{-2} + .. \; e_{n_{ej}} q^{-ne}$$

$$F_j(q^{-1}) = f_0 + f_1 q^{-1} + f_2 q^{-2} + .. \; f_{n_{fj}} q^{-nfj}$$

Another useful formulation

$$C(q^{-1}) = A(q^{-1})D(q^{-1})E_j(q^{-1}) + q^{-j}F_j(q^{-1})$$

Remark ***Simular to the diophantine equation (useful for Matlab / Simulink implementation)***

A polynomial division

At which rank to we decide to stop the division ? *nej = j -1*

$$\frac{C(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j) = E_j(q^{-1})\gamma(t+j) + q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j)$$

$$E_j(q^{-1})\gamma(t+j) = e_0\gamma(t+j) + e_1\gamma(t+j-1)+..+e_{j-1}\gamma(t+1)$$ *Unpredictible part*

$$q^{-j}\frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t+j) = \frac{F_j(q^{-1})}{A(q^{-1})D(q^{-1})}\gamma(t)$$ *Avalaible part*

$$nej = \mathrm{j} - 1$$

$$nef \leq \max(n_a + n_d - 1, n_c + j)$$

Using the polynomial division

$$A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})u(t+j) + v(t+j)$$

$$\downarrow$$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})D(q^{-1})u(t+j) + D(q^{-1})\, v(t+j)$$

$$\downarrow$$

$$D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})D(q^{-1})u(t+j) + C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$E(q^{-1})D(q^{-1})A(q^{-1})y(t+j) = q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j) + E(q^{-1})\, C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$\{C(q^{-1}) - q^{-j}F(q^{-1})\}y(t+j) = q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j) + E(q^{-1})\, C(q^{-1})\gamma(t+j)$$

$$\downarrow$$

$$C(q^{-1})y(t+j) = q^{-j}F(q^{-1})y(t+j) + q^{-d-1}B(q^{-1})E(q^{-1})D(q^{-1})u(t+j) + E(q^{-1})\, C(q^{-1})\gamma(t+j)$$

Using the polynomial division

$$y(t+j)=\frac{F(q^{-1})}{C(q^{-1})}y(t)+\frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)+E(q^{-1})\,\gamma(t+j)$$

unpredictible

The optimal prediction

$$\hat{y}(t+j/t)=\frac{F(q^{-1})}{C(q^{-1})}y(t)+\frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

Is the prediction optimal ?

$$\tilde{y}(t+j/t) = y(t+j) - \hat{y}(t+j/t)$$

$$\tilde{y}(t+j/t) = E(q^{-1})\,\gamma(t+j) = e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)$$

Mean value

$$\varepsilon\{\tilde{y}(t+j/t)\} = \{e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)\}$$
$$= \varepsilon\left(\sum_{k=0}^{j-1}\{e_k\gamma(t+j-k)\}\right)$$
$$= \sum_{k=0}^{j-1}\{e_k\varepsilon(\gamma(t+j-k))\}$$
$$=0$$

Is the prediction optimal ?

$$\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\} = \varepsilon\left\{\left(e_0\gamma(t+j) + e_1\gamma(t+j-1) + .. + e_{j-1}\gamma(t+1)\right)^2\right\}$$

$$= \varepsilon\left\{\sum_{i=0}^{j-1}\sum_{k=0}^{j-1} e_i e_k \gamma(t+j-i)\gamma(t+j-k)\right\}$$

$$= \sum_{i=0}^{j-1} {e_i}^2\,\varepsilon\{(\gamma(t))^2\}$$

$$= \sum_{i=0}^{j-1} {e_i}^2\,\sigma^2$$

Conclusion

No bias

- $\varepsilon\{\tilde{y}(t+j/t)\} = 0$

Minimal Variance

$$\varepsilon\left\{\left(\tilde{y}(t+j/t)\right)^2\right\} = \sum_{i=0}^{j-1} {e_i}^2\,\sigma^2$$

Prediction dynamics imposed by $C(q^{-1})$

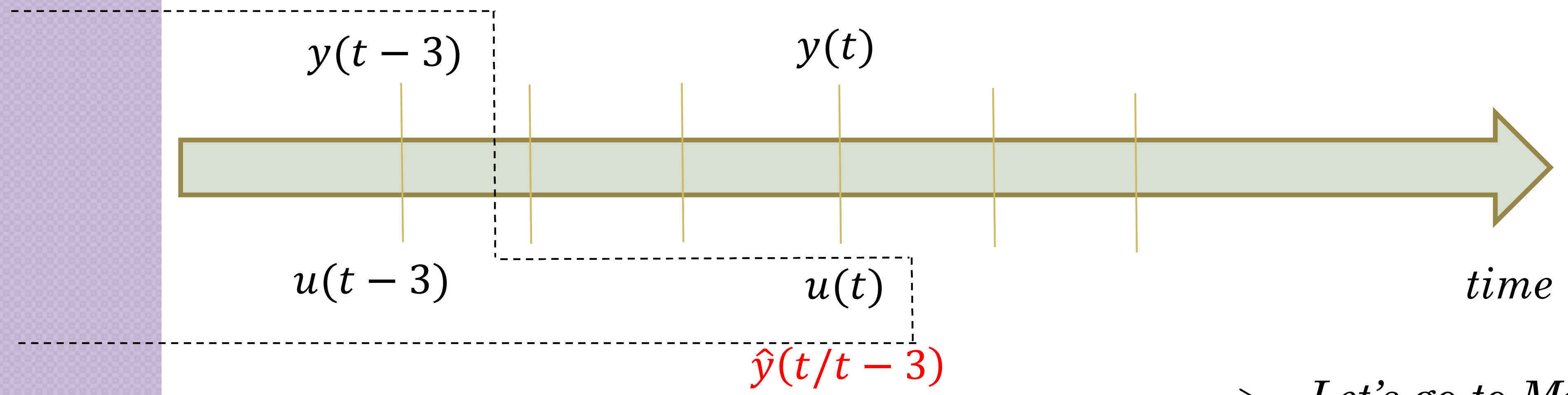
$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

The optimal prediction ***is not causal***

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

→ *Depends on the future values of the control variable*
Can not be simulated using Simulink

We are going to simulate $\hat{y}(t/t-j)$



→ *Let's go to Matlab*

Predictive control

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Outline

1 - Minimal variance control

$$J(u(t)) = \varepsilon\left(\left(y(t+d+1) - y^*(t+d+1)\right)^2\right)$$

2 - One step ahead predictive control

$$J(u(t)) = \varepsilon\left(\left(y(t+d+1) - y^*(t+d+1)\right)^2 + \mu(D(q^{-1})u(t))^2\right)$$

3 - One step ahead predictive control with frequency weighting

$$J(u(t)) = \varepsilon\left(\left(y(t+d+1) - y^*(t+d+1)\right)^2 + \mu\left(u_f(t)\right)^2\right)$$

$$u_f(t) = \frac{W(q^{-1})}{H(q^{-1})}u(t)$$

Minimal Variance Control

The criteria

Find the control value u(t) that minimizes the following criteria

$$J(u(t)) = \varepsilon\left(\left(y(t+d+1) - y^*(t+d+1)\right)^2\right)$$

Optimal prediction

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t+j-d-1)$$

$$\Big\downarrow \; j = d+1$$

$$\hat{y}(t+d+1/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t)$$

$$C(q^{-1}) = A(q^{-1})D(q^{-1})E(q^{-1}) + q^{-d-1}F(q^{-1})$$

Derivation of the criteria

$$J(u(t)) = \varepsilon\left(\left(y(t+d+1) - y^*(t+d+1)\right)^2\right)$$

$$\hat{J}(u(t)) = \varepsilon\left(\left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right)^2\right)$$

Derivation

$$\frac{\partial \hat{J}(u(t))}{\partial u(t)} = 2\left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right)\frac{\partial \hat{y}(t+d+1/t)}{\partial u(t)}$$

$$\frac{\partial \hat{y}(t+d+1/t)}{\partial u(t)} = \frac{b_0 e_0 d_0}{c_0} = \frac{b_0 e_0}{c_0} = b_0$$

$$\frac{\partial \hat{J}(u(t))}{\partial u(t)} = 0 \Rightarrow \left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right) = 0$$

Linear Time Invariant controller structure

$$\frac{\partial \hat{J}\,(u(t))}{\partial u(t)} = 0 \Rightarrow \left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right) = 0$$

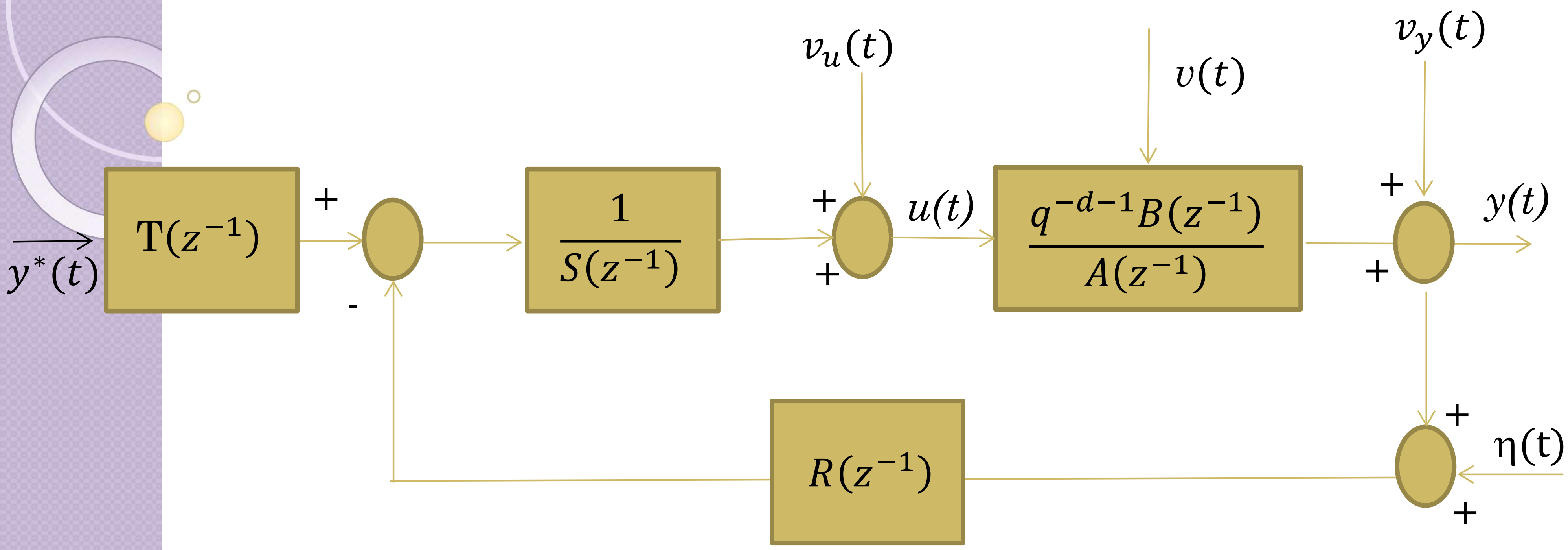
$$\Rightarrow \quad \hat{y}(t+d+1/t) = y^*(t+d+1)$$

$$\Rightarrow \quad \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E(q^{-1})D(q^{-1})}{C(q^{-1})}u(t) = y^*(t+d+1)$$

$$\Rightarrow \quad F(q^{-1})y(t) + B(q^{-1})E(q^{-1})D(q^{-1})u(t) = C(q^{-1})y^*(t+d+1)$$

$$\Rightarrow \quad \boxed{R(q^{-1})y(t) + S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1)}$$

Closed-loop analysis



$v_u(t)$ *Input disturbance (low frequency)*

$v_y(t)$ *Output disturbance (low frequency)*

$y^*(t)$ *Reference sequence*

$\eta(t)$ *Noise mesurements (high frequency)*

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Closed-loop performances

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})\big(u(t)\big) + v(t)$$ *System equation*

$$R(q^{-1})y(t) + S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1)$$ *Controller equation*

Output performances

$$y(t) = \frac{B(q^{-1})T(q^{-1})}{P_c(q^{-1})}y^*(t) + \frac{S(q^{-1})}{P_c(q^{-1})}v(t)$$

Input performances

$$u(t) = \frac{A(q^{-1})T(q^{-1})}{P_c(q^{-1})}y^*(t+d+1) - \frac{R(q^{-1})}{P_c(q^{-1})}v(t)$$

Characteristic polynomial

$$
\begin{aligned}
P_c(z^{-1}) &= A(z^{-1})S(z^{-1}) + q^{-d-1}B(z^{-1})R(z^{-1}) \\
&= A(z^{-1})E(z^{-1})B(z^{-1})D(z^{-1}) + q^{-d-1}B(z^{-1})F(z^{-1}) \\
&= B(z^{-1})\left(A(z^{-1})E(z^{-1})D(z^{-1}) + q^{-d-1}F(z^{-1})\right) \\
&= B(z^{-1})C(z^{-1})
\end{aligned}
$$

!

$B(z^{-1})$ *and* $\mathrm{C}(z^{-1})$ *MUST be stable polynomials (Hurwitz)*

Output tracking performances

$$y(t) = \frac{B(q^{-1})T(q^{-1})}{P_c(q^{-1})} y^*(t) = \frac{B(q^{-1})C(q^{-1})}{B(q^{-1})C(q^{-1})} y^*(t) = y^*(t)$$

Perfect tracking !!

Disturbance rejection performances

$$y(t) = \frac{S(q^{-1})}{P_c(q^{-1})} v(t) = \frac{E(q^{-1})B(q^{-1})D(q^{-1})}{B(q^{-1})C(q^{-1})} v(t) = \frac{E(q^{-1})D(q^{-1})}{C(q^{-1})} v(t) = E(q^{-1})\gamma(t)$$

$$y(t) - y^*(t) = E(q^{-1})\gamma(t) = \tilde{y}\,(t/t-d-1)$$

Minimal Variance Control

Closed-loop performances

Input tracking performances

$$\mathrm{u}(t) = \frac{A(q^{-1})T(q^{-1})}{P_c(q^{-1})} y^*(t+d+1) = \frac{A(q^{-1})}{B(q^{-1})} y^*(t+d+1)$$

➡ *Inversion of the model!!*

➡ *High Energy consumption and input saturation problem*

Input rejection performances

➡ $$u(t) = -\frac{R(q^{-1})}{P_c(z^{-1})} v(t) = -\frac{F(q^{-1})}{B(z^{-1})C(z^{-1})} v(t)$$

One step ahead predictive control

The modified criteria

Find the control value u(t) that minimizes the following criteria

$$J(u(t)) = \varepsilon\left(\left(y(t+d+1) - y^*(t+d+1)\right)^2 + \mu\left(D(q^{-1})u(t)\right)^2\right)$$

Additionnal term $\triangleq$ *energy consumption term*

$\mu = 0 \Rightarrow$ *Minimal Variance Control*

$$\hat{J}(u(t)) = \left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right)^2 + \mu\left(D(q^{-1})u(t)\right)^2$$

Derivation of the criteria

$$\hat{J}(u(t)) = \left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right)^2 + \mu(D(q^{-1})u(t))^2$$

$$\frac{\partial \hat{J}(u(t))}{\partial u(t)} = 2\left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right)\frac{\partial \hat{y}(t+d+1/t)}{\partial u(t)}$$
$$+2\mu D(q^{-1})u(t)\frac{\partial(D(q^{-1})u(t))}{\partial u(t)}$$

$$\frac{\partial \hat{y}(t+d+1/t)}{\partial u(t)} = \frac{b_0 e_0 d_0}{c_0} = b_0$$

$$\frac{\partial(D(q^{-1})u(t))}{\partial u(t)} = \frac{\partial(u(t) + d_1 u(t-1) + .. + d_{nd} u(t-n_d))}{\partial u(t)} = 1$$

Derivation of the criteria

$$\frac{\partial \hat{J}(u(t))}{\partial u(t)} = 2b_0\big(\hat{y}(t+d+1/t) - y^*(t+d+1)\big) + 2\mu D(q^{-1})u(t)$$

$$\frac{\partial \hat{J}(u(t))}{\partial u(t)} = 0 \Rightarrow D(q^{-1})u(t) = \frac{b_0}{\mu}\big(y^*(t+d+1) - \hat{y}(t+d+1/t)\big)$$

Let us introduce the prediction equation to replace $\hat{y}(t+d+1/t)$

$$\xrightarrow[C(q^{-1})]{\text{Operate by}} \quad C(q^{-1})D(q^{-1})u(t) = \frac{b_0}{\mu}\big(C(q^{-1})y^*(t+d+1) - C(q^{-1})\hat{y}(t+d+1/t)\big)$$

Linear Time Invariant controller structure

$$C(q^{-1})D(q^{-1})u(t) = \frac{b_0}{\mu}\left(C(q^{-1})y^*(t+d+1) - C(q^{-1})\hat{y}(t+d+1/t)\right)$$

Introduce the prediction equation

$$C(q^{-1})D(q^{-1})u(t) = \frac{b_0}{\mu}(C(q^{-1})y^*(t+d+1) - F(q^{-1})y(t) - E(q^{-1})B(q^{-1})D(q^{-1})u(t))$$

$$\left\{\frac{b_0}{\mu}E(q^{-1})B(q^{-1})D(q^{-1}) + C(q^{-1})D(q^{-1})\right\}u(t) + \frac{b_0}{\mu}F(q^{-1})y(t) = \frac{b_0}{\mu}C(q^{-1})y^*(t+d+1)$$

$$\boxed{S(q^{-1})u(t) + R(q^{-1})y(t) = T(q^{-1})y^*(t+d+1)}$$

Closed-loop performances

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})\big(u(t)\big) + v(t)$$ *System equation*

$$R(q^{-1})y(t) + S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1)$$ *Controller equation*

Output performances

$$y(t) = \frac{B(q^{-1})T(q^{-1})}{P_c(q^{-1})}y^*(t) + \frac{S(q^{-1})}{P_c(q^{-1})}v(t)$$

Input performances

$$u(t) = \frac{A(q^{-1})T(q^{-1})}{P_c(q^{-1})}y^*(t+d+1) - \frac{R(q^{-1})}{P_c(q^{-1})}v(t)$$

Characteristic polynomial

$$P_c(z^{-1}) = A(z^{-1})S(z^{-1}) + q^{-d-1}B(z^{-1})R(z^{-1})$$

$$= A(z^{-1})\left\{\frac{b_0}{\mu}E(z^{-1})B(z^{-1})D(z^{-1}) + C(z^{-1})D(z^{-1})\right\}$$

$$+ q^{-d-1}B(z^{-1})\frac{b_0}{\mu}F(z^{-1})$$

$$= A(z^{-1})C(z^{-1})D(z^{-1}) + \frac{b_0}{\mu}B(z^{-1})\{A(z^{-1})E(z^{-1})D(z^{-1}) + q^{-d-1}F(z^{-1})\}$$

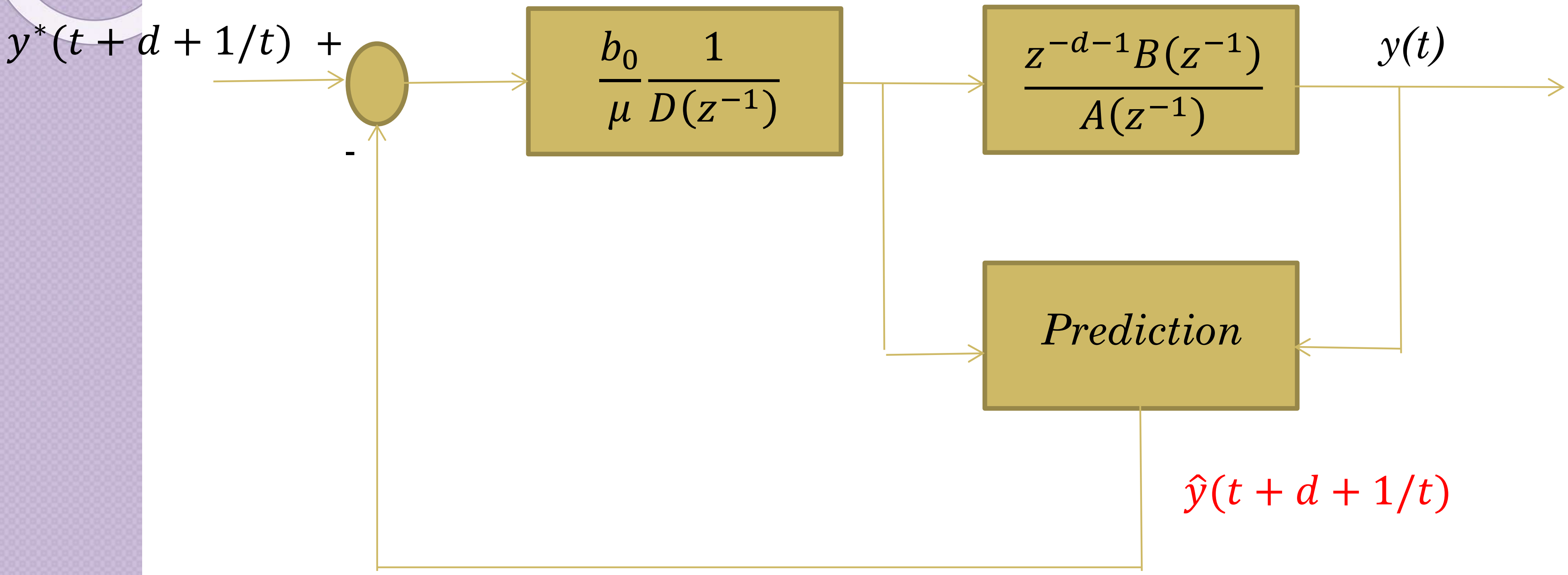
Introduce the prediction equation

$$P_c(z^{-1}) = A(z^{-1})C(z^{-1})D(z^{-1}) + \frac{b_0}{\mu}B(z^{-1})C(z^{-1})$$

$$P_c(z^{-1}) = C(z^{-1})\left\{A(z^{-1})D(z^{-1}) + \frac{b_0}{\mu}B(z^{-1})\right\}$$

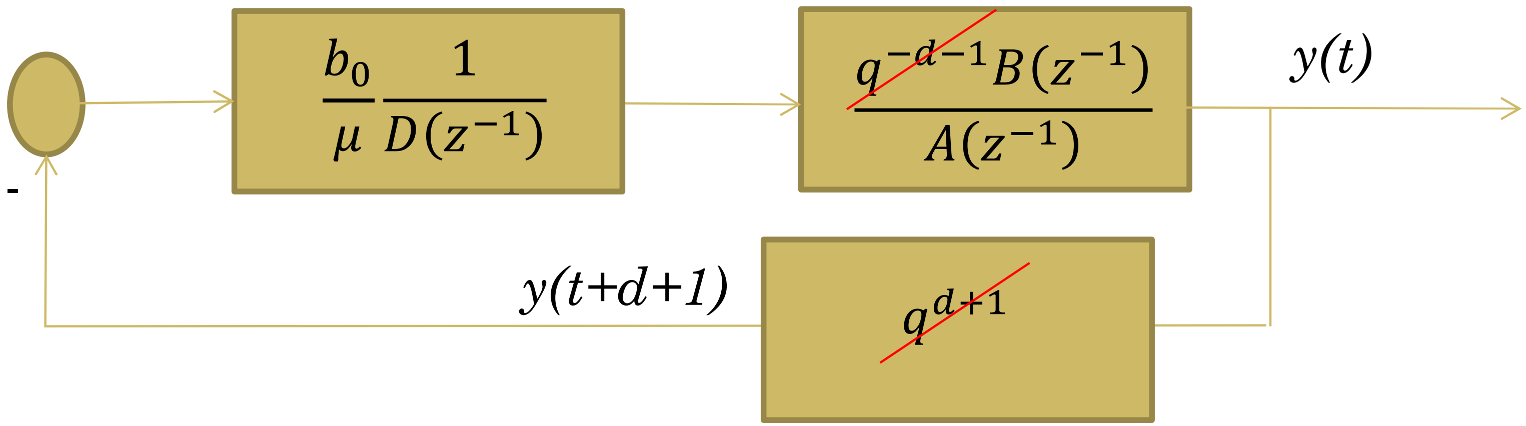
Equivalent implementation scheme

$$D(q^{-1})u(t) = \frac{b_0}{\mu}\left(y^*(t+d+1) - \hat{y}(t+d+1/t)\right)$$



Equivalent scheme for stability analysis

$$v(t) = 0 \Rightarrow \hat{y}(t+d+1/t) = y(t+d+1/t) = q^{d+1}y(t)$$



$$P_c(z^{-1}) = C(z^{-1})\left\{A(z^{-1})D(z^{-1}) + \frac{b_0}{\mu}B(z^{-1})\right\} \Rightarrow P_c(z^{-1}) = A(z^{-1})D(z^{-1}) + \frac{b_0}{\mu}B(z^{-1})$$

A single synthesis parameter : root-locus tool

Tracking performances

$$y(t)=\frac{B(q^{-1})T(q^{-1})}{P_c(q^{-1})}y^*(t)=\frac{\frac{b_0}{\mu}B(q^{-1})C(q^{-1})}{C(q^{-1})P_f(q^{-1})}y^*(t)=\frac{b_0}{\mu}\frac{B(q^{-1})}{P_f(q^{-1})}y^*(t)$$

$$\Rightarrow \quad \frac{y(z^{-1})}{y^*(z^{-1})}=\frac{b_0}{\mu}\frac{B(z^{-1})}{P_f(z^{-1})}=\frac{b_0}{\mu}\frac{B(z^{-1})}{A(z^{-1})D(z^{-1})+\frac{b_0}{\mu}B(z^{-1})}$$

$\Rightarrow$ *Tracking dynamics does not depend on the predictor*

Static performances

No bias $\Rightarrow$

$$\frac{b_0}{\mu}\frac{B(1)}{A(1)D(1)+\frac{b_0}{\mu}B(1)}=1\Rightarrow D(1)=0$$

$$\Rightarrow D(q^{-1})=(1-q^{-1})\mathrm{D}'(q^{-1})$$

Integral action

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Disturbance rejection

$$y(t) = \frac{S(q^{-1})}{P_c(q^{-1})} v(t) \qquad \mathrm{S}(q^{-1}) = D(q^{-1})\left(\frac{b_0}{\mu} E(q^{-1}) B(q^{-1}) + \mathrm{C}(q^{-1})\right)$$

$$y(t) = \frac{D(q^{-1})\left(\frac{b_0}{\mu} E(q^{-1}) B(q^{-1}) + \mathrm{C}(q^{-1})\right)}{C(q^{-1}) P_f(q^{-1})} v(t)$$

$$= \frac{D(q^{-1})\left(\frac{b_0}{\mu} E(q^{-1}) B(q^{-1}) + \mathrm{C}(q^{-1})\right)}{C(q^{-1}) P_f(q^{-1})} \frac{C(q^{-1})}{D(q^{-1})} \gamma(t)$$

$$= \frac{\left(\frac{b_0}{\mu} E(q^{-1}) B(q^{-1}) + \mathrm{C}(q^{-1})\right)}{P_f(q^{-1})} \gamma(t)$$

$P_f(q^{-1})$ *Hurwitz* ⇒ *Disturbance rejection*

One step ahead predictive control with input frequency weighting

The modified criteria

Find the control value u(t) that minimizes the following criteria

$$J(u(t)) = \varepsilon\left(\left(y(t+d+1) - y^*(t+d+1)\right)^2 + \mu\left(u_f(t)\right)^2\right)$$

with $u_f(t) = \frac{W(q^{-1})}{H(q^{-1})}u(t)$ $\mu = \frac{b_0 h_0}{w_0}$

$\frac{W(q^{-1})}{H(q^{-1})}$ *is called Input Frequency Weighting*

Derivation of the criteria

$$\hat{J}(u(t)) = \left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right)^2 + \mu\left(u_f(t)\right)^2$$

$$\frac{\partial \hat{J}(u(t))}{\partial u(t)} = 2\left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right)\frac{\partial \hat{y}(t+d+1/t)}{\partial u(t)}$$

$$+2\mu u_f\,(t)\frac{\partial\left(u_f\,(t)\right)}{\partial u(t)}$$

$$\frac{\partial \hat{y}(t+d+1/t)}{\partial u(t)} = \frac{b_0 e_0 d_0}{c_0} = b_0$$

$$\frac{\partial\left(u_f(t)\right)}{\partial u(t)} = \frac{w_0}{h_0}$$

Derivation of the criteria

$$\frac{\partial \hat{J}(u(t))}{\partial u(t)} = 2b_0\left(\hat{y}(t+d+1/t) - y^*(t+d+1)\right) + 2\mu\frac{w_0}{h_0}u_f(t)$$

$$\frac{\partial \hat{J}(u(t))}{\partial u(t)} = 0 \Rightarrow u_f(t) = \frac{b_0 h_0}{\mu w_0}\left(y^*(t+d+1) - \hat{y}(t+d+1/t)\right)$$

Let us introduce the prediction equation to replace $\hat{y}(t+d+1/t)$

$$\xrightarrow[\mathrm{C}(q^{-1})]{\text{Operate by}} \quad \mathrm{C}(q^{-1})u_f(t) = \left(C(q^{-1})y^*(t+d+1) - C(q^{-1})\hat{y}(t+d+1/t)\right)$$

Linear Time Invariant controller

$$C(q^{-1})u_f(t) = \{C(q^{-1})y^*(t+d+1) - F(q^{-1})y(t) - E(q^{-1})B(q^{-1})D(q^{-1})u(t)\}$$

Let us introduce the relation between $u_f(t)$ *and* $u(t)$

$$\xrightarrow[\mathrm{H}(q^{-1})]{\text{Operate by}} \quad C(q^{-1})H(q^{-1})u_f(t) = H(q^{-1})\{C(q^{-1})y^*(t+d+1) - F(q^{-1})y(t) - E(q^{-1})B(q^{-1})D(q^{-1})u(t)\}$$

$$C(q^{-1})W(q^{-1})u(t) = H(q^{-1})\{C(q^{-1})y^*(t+d+1) - F(q^{-1})y(t) - E(q^{-1})B(q^{-1})D(q^{-1})u(t)\}$$

$$\{C(q^{-1})W(q^{-1}) + H(q^{-1})E(q^{-1})B(q^{-1})D(q^{-1})\}u(t) + H(q^{-1})\,F(q^{-1})y(t) = H(q^{-1})C(q^{-1})y^*(t+d+1)$$

$$\boxed{S(q^{-1})u(t) + R(q^{-1})y(t) = T(q^{-1})y^*(t+d+1)}$$

Disturbance rejection

$$y(t) = \frac{S(q^{-1})}{P_c(q^{-1})} v(t) \qquad S(q^{-1}) = C(q^{-1})W(q^{-1}) + H(q^{-1})E(q^{-1})B(q^{-1})D(q^{-1})$$

$$y(t) = \frac{C(q^{-1})W(q^{-1}) + H(q^{-1})E(q^{-1})B(q^{-1})D(q^{-1})}{P_c(q^{-1})} v(t)$$

$$= \frac{C(q^{-1})W(q^{-1}) + H(q^{-1})E(q^{-1})B(q^{-1})D(q^{-1})}{P_c(q^{-1})} \frac{C(q^{-1})}{D(q^{-1})} \gamma(t)$$

$P_c(q^{-1})$ *Hurwitz but* $D(q^{-1})$ *is not a Hurwitz polynomial*

Disturbance rejection $\Leftrightarrow$ $D(q^{-1})$ *factorizes* $S(q^{-1})$

$$\Leftrightarrow \quad W(q^{-1}) = D(q^{-1})G(q^{-1})$$

$$S(q^{-1}) = D(q^{-1})\{C(q^{-1})G(q^{-1}) + H(q^{-1})E(q^{-1})B(q^{-1})\}$$

Closed-loop performances

$$A(q^{-1})y(t) = q^{-d-1}B(q^{-1})\big(u(t)\big) + v(t)$$ *System equation*

$$R(q^{-1})y(t) + S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1)$$ *Controller equation*

Output performances

$$y(t) = \frac{B(q^{-1})T(q^{-1})}{P_c(q^{-1})}y^*(t) + \frac{S(q^{-1})}{P_c(q^{-1})}v(t)$$

Input performances

$$u(t) = \frac{A(q^{-1})T(q^{-1})}{P_c(q^{-1})}y^*(t+d+1) - \frac{R(q^{-1})}{P_c(q^{-1})}v(t)$$

Closed-loop performances

Characteristic polynomial

$$S(q^{-1}) = D(q^{-1})\{C(q^{-1})G(q^{-1}) + H(q^{-1})E(q^{-1})B(q^{-1})\}$$

$$P_c(q^{-1}) = A(q^{-1})S(q^{-1}) + q^{-d-1}B(q^{-1})R(q^{-1})$$

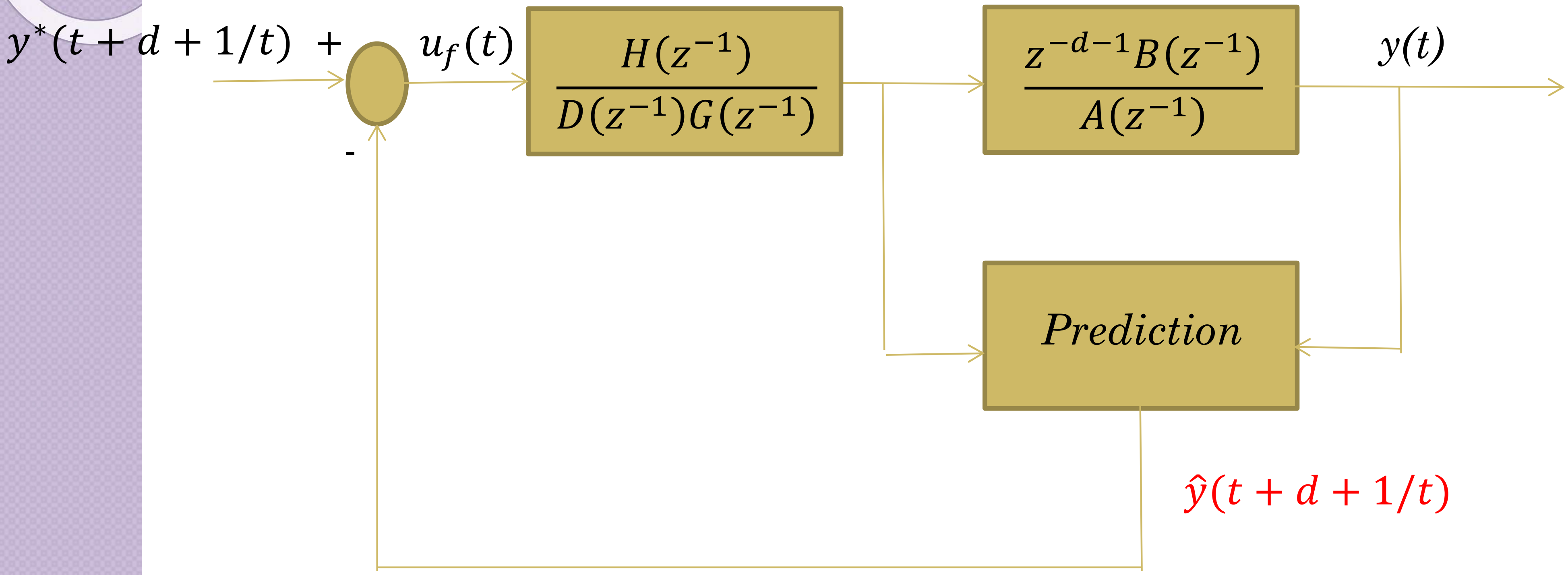
$$= A(q^{-1})D(q^{-1})\{C(q^{-1})G(q^{-1}) + H(q^{-1})E(q^{-1})B(q^{-1})\} + q^{-d-1}B(q^{-1})H(q^{-1})F(q^{-1})$$

$$= A(q^{-1})C(q^{-1})D(q^{-1})G(q^{-1}) + B(q^{-1})H(q^{-1})\{A(q^{-1})E(q^{-1})D(q^{-1}) + q^{-d-1}F(q^{-1})\}$$

$$\Rightarrow \quad P_c(q^{-1}) = C(q^{-1})\{A(q^{-1})D(q^{-1})G(q^{-1}) + B(q^{-1})H(q^{-1})\}$$

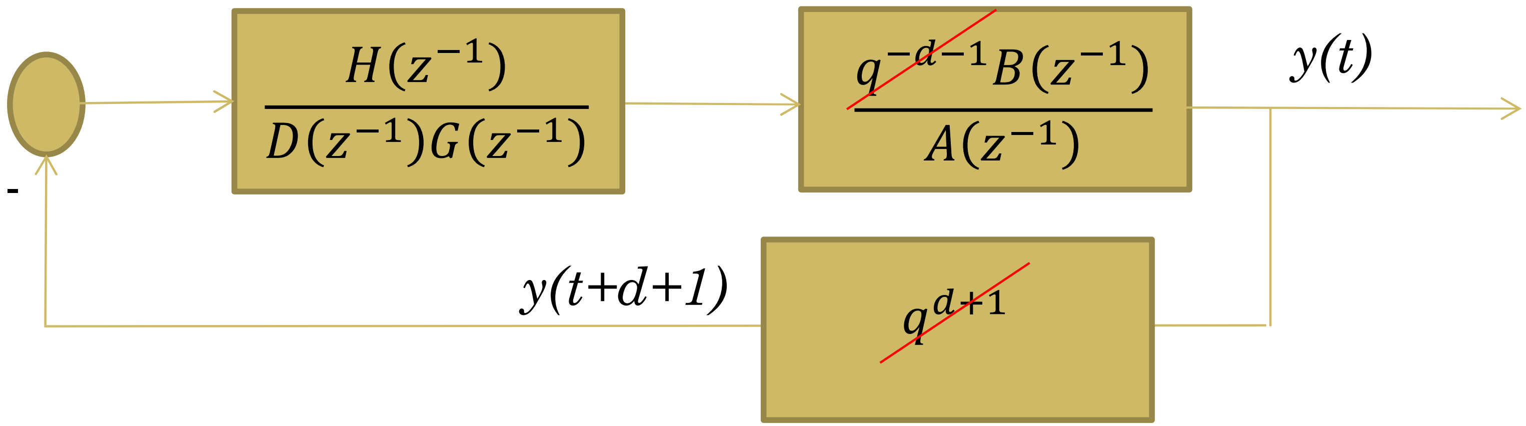
Equivalent implementation scheme

$$u_f(t) = \left(y^*(t+d+1) - \hat{y}(t+d+1/t)\right)$$



Equivalent scheme for stability analysis

$$v(t) = 0 \Rightarrow \hat{y}(t+d+1/t) = y(t+d+1/t) = q^{d+1}y(t)$$

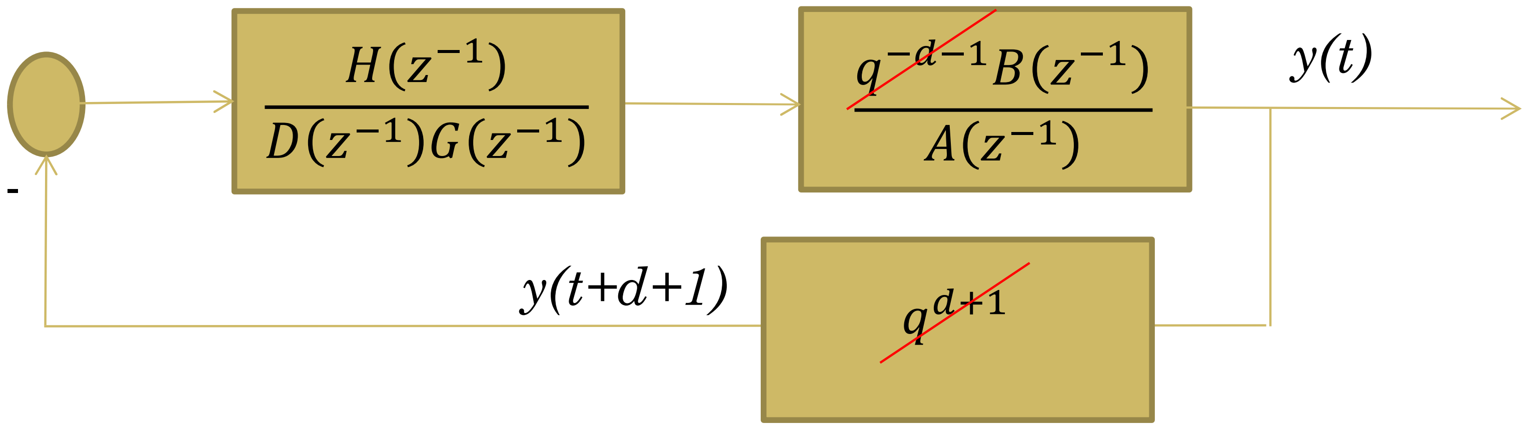


$$P_c(z^{-1}) = C(z^{-1})\{A(z^{-1})D(z^{-1})G(z^{-1}) + B(z^{-1})H(z^{-1})\}$$

Frequency weighting synthesis : pole placement or frequency design

Equivalent scheme for stability analysis

$$v(t) = 0 \Rightarrow \hat{y}(t+d+1/t) = y(t+d+1/t) = q^{d+1}y(t)$$



$$P_c(z^{-1}) = C(z^{-1})\{A(z^{-1})D(z^{-1})G(z^{-1}) + B(z^{-1})H(z^{-1})\}$$

Frequency weighting synthesis : pole placement or frequency design

$$y(t) = \frac{B(q^{-1})T(q^{-1})}{P_c(q^{-1})} y^*(t) = \frac{B(q^{-1})H(q^{-1})C(q^{-1})}{C(q^{-1})(A(z^{-1})D(z^{-1})G(z^{-1}) + B(z^{-1})H(z^{-1}))} y^*(t)$$

$$\Rightarrow \quad \frac{y(z^{-1})}{y^*(z^{-1})} = \frac{B(z^{-1})H(z^{-1})}{A(z^{-1})D(z^{-1})G(z^{-1}) + B(z^{-1})H(z^{-1})}$$

$\Rightarrow$ *Tracking dynamics does not depend on the predictor*

Static performances

No bias $\Rightarrow$

$$\frac{B(1)H(1)}{A(1)D(1)G(1) + B(1)H(1)} = 0 \Rightarrow D(1) = 0$$

$$D(q^{-1}) = (1 - q^{-1})D'(q^{-1})$$

Integral action

If one choses $T(q^{-1})$ *such that*

$$T(q^{-1}) = \frac{1}{B(1)} P_c(q^{-1})$$

$$\frac{y(z^{-1})}{y^*(z^{-1})} = \frac{B(z^{-1})T(z^{-1})}{P_c(z^{-1})} = \frac{B(z^{-1})}{B(1)}$$ ⇒ *Semi-perfect tracking*

Moreover, if one choses $T(q^{-1})$ *such that*

$$A(z^{-1})D(z^{-1})G(z^{-1}) + B(z^{-1})H(z^{-1}) = B(z^{-1})M(z^{-1})$$

$$\mathrm{T}(z^{-1}) = M(z^{-1})$$

$$\frac{y(z^{-1})}{y^*(z^{-1})} = \frac{B(z^{-1})T(z^{-1})}{P_c(z^{-1})} = \frac{B(z^{-1})M(z^{-1})}{B(z^{-1})M(z^{-1})} = 1$$ ⇒ *Perfect tracking*

! *Perfect tracking IF AND ONLY IF* $\mathrm{B}(q^{-1})$ *HURWITZ*

Disturbance rejection

$$y(t)=\frac{S(q^{-1})}{P_c(q^{-1})}v(t) \qquad \mathrm{S}(q^{-1})=D(q^{-1})\left(\frac{b_0}{\mu}E(q^{-1})B(q^{-1})+\mathrm{C}(q^{-1})\right)$$

$$\Rightarrow\ y(t)=\frac{D(q^{-1})\left(\frac{b_0}{\mu}E(q^{-1})B(q^{-1})+\mathrm{C}(q^{-1})\right)}{C(q^{-1})P_f(q^{-1})}v(t)$$

$$=\frac{D(q^{-1})\left(\frac{b_0}{\mu}E(q^{-1})B(q^{-1})+\mathrm{C}(q^{-1})\right)}{C(q^{-1})P_f(q^{-1})}\frac{C(q^{-1})}{D(q^{-1})}\gamma(t)$$

$$=\frac{\left(\frac{b_0}{\mu}E(q^{-1})B(q^{-1})+\mathrm{C}(q^{-1})\right)}{P_f(q^{-1})}\gamma(t)$$

$\Rightarrow$ $P_f(q^{-1})$ *Hurwitz* $\Rightarrow$ *Disturbance rejection*

Exercice on Matlab / Simulink

Predictive control

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Schedule

1 – Optimal prediction for control design purposes

2 – Criteria and derivation of the criteria

3 – Linear Time Invariant controller

4 – Input / Output performances

Optimal prediction for control design purposes

New parametrization of the predictor

$$y(t+j) = \frac{F_j(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E_j(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1) + E_j(q^{-1})\gamma(t+j)$$

Let's have a look at $\frac{B(q^{-1})E_j(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1)$

It contains terms $\{u(t+j-d-1) \quad \ldots \quad u(t)\}$ *and* $\{u(t-1) \quad u(t-2) \quad ..\}$

Future control values that have to be calculated

Already available at time t

New parametrization of the predictor

We are going to separate the future and the past values of the control values

We can do that with a second polynomial division

$$\frac{B(q^{-1})E_j(q^{-1})}{C(q^{-1})} = G_{j-d}(q^{-1}) + q^{-j+d}\frac{H_{j-d}(q^{-1})}{C(q^{-1})}$$

$$G(q^{-1}) = g_0 + g_1 q^{-1} + \cdots g_{j-d-1} q^{-j+d+1}$$

$$H(q^{-1}) = h_0 + h_1 q^{-1} + \cdots h_{nh} q^{-nh}$$

!

The degree (j-d-1) of $G(q^{-1})$ plays an important role

Hence

$$\frac{B(q^{-1})E_j(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1)$$

$$= G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + q^{-j+d}\frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1)$$

$$= \left(g_0 + g_1 q^{-1} + .. + g_{j-d-1}q^{-j+d+1}\right)D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)$$

$$= \underbrace{g_0 D(q^{-1})u(t+j-d-1) + .. + g_{j-d-1}D(q^{-1})u(t)}_{\text{Future control values (to be calculated)}} + \underbrace{\frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)}_{\text{Last control values (already known)}}$$

New parametrization of the predictor

$$y(t+j) = \frac{F_j(q^{-1})}{C(q^{-1})}y(t) + \frac{B(q^{-1})E_j(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1) + E_j(q^{-1})\gamma(t+j)$$

$$y(t+j) = \frac{F_j(q^{-1})}{C(q^{-1})}y(t) + G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1) + E_j(q^{-1})\gamma(t+j)$$

To summurize

$$G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1)$$

Only depends on the actual and future control values

$$\frac{F_j(q^{-1})}{C(q^{-1})}y(t)+\frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)$$

Is completely known at time t

$$E_j(q^{-1})\gamma(t+j)$$

Is unpredictible

The optimal j-step predictor

$$y(t+j) = \frac{F_j(q^{-1})}{C(q^{-1})} y(t) + G_{j-d}(q^{-1}) D(q^{-1}) u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})} D(q^{-1}) u(t-1) + E_j(q^{-1}) \gamma(t+j)$$

What is avalaible for prediction ?

$$\hat{y}(t+j/t) = \frac{F_j(q^{-1})}{C(q^{-1})} y(t) + G_{j-d}(q^{-1}) D(q^{-1}) u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})} D(q^{-1}) u(t-1)$$

$$\hat{y}(t+j/t) = G_{j-d}(q^{-1}) D(q^{-1}) u(t+j-d-1) + \hat{y}_0(t+j/t)$$

with

$$\hat{y}_0(t+j/t) = \frac{F_j(q^{-1})}{C(q^{-1})} y(t) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})} D(q^{-1}) u(t-1)$$

A set of predictors

$$\hat{y}(t+j/t) = G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \hat{y}_0(t+j/t)$$

From j = d+1 to j = h_p

$$\hat{y}(t+d+1/t) = g_0 D(q^{-1})u(t) + \hat{y}_0(t+d+1/t)$$

$$\hat{y}(t+d+2/t) = g_0 D(q^{-1})u(t+1) + g_1 D(q^{-1})u(t) + \hat{y}_0(t+d+2/t)$$

$$\hat{y}(t+h_p/t) = g_0 D(q^{-1})u(t+h_p-d-1) + .. g_{h_p-d-1} D(q^{-1})u(t) + \hat{y}_0(t+h_p/t)$$

Matricial Form of the set of predictors

$$\begin{pmatrix} \hat{y}(t+d+1/t) \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+d+1/t) \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

$$\hat{Y}(t+h_p/t) = \Phi D(q^{-1}) U(t+h_p-d-1) + \hat{Y}_0(t+h_p/t)$$

Criteria and derivation of the criteria

The criteria

- *Find the control vector* $U(t+h_c-1)$ *that minimizes*

$$J(U(t+h_c-1)\ ) = \varepsilon\left(\sum_{j=h_i}^{h_p}\{y(t+j)-y^*(t+j)\}^2\right)+\varepsilon\left(\sum_{j=h_i}^{h_p}\lambda\{D(q^{-1})u(t+j-h_i))\}^2\right)$$

with $U(t+h_c-d-1) = [u(t) \quad \dots \quad u(t+h_c-1)]$

$\mathrm{u}(t+j) = 0 \quad \forall j \geq h_c$

An equivalent criteria

- *The control vector* $U(t+h_c-1)$ *that minimizes*

$$J(U(t+h_c-1)\ ) = \varepsilon\left(\sum_{j=h_i}^{h_p}\{y(t+j)-y^*(t+j)\}^2\right) + \varepsilon\left(\sum_{j=h_i}^{h_p}\lambda\{D(q^{-1})u(t+j-h_i))\}^2\right)$$

Also minimizes

$$\hat{J}(U(t+h_c-1)\ ) = \sum_{j=h_i}^{h_p}\{\hat{y}(t+j/t)-y^*(t+j)\}^2 + \sum_{j=h_i}^{h_p}\lambda\{D(q^{-1})u(t+j-h_i))\}^2$$

New matricial expression for the set of predictors

$$\begin{pmatrix} \hat{y}(t+d+1/t) \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+d+1/t) \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

1 *We need the predictors from* h_i *to* h_p

We suppress the first lines

$$\begin{pmatrix} \cancel{\hat{y}(t+d+1/t)} \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} \cancel{g_0} & \cancel{\cdots} & \cancel{0} \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \cancel{\hat{y}_0(t+d+1/t)} \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

2 *Control values are nul if* $j \geq h_c$

We suppress the last columns of Φ

$$\begin{pmatrix} \cancel{\hat{y}(t+d+1/t)} \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} \cancel{g_0} & \cancel{\cdots} & \cancel{0} \\ \vdots & \ddots & \cancel{\vdots} \\ g_{h_p-d-1} & \cdots & \cancel{g_0} \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ \cancel{u(t+h_p-d-1)} \end{pmatrix} + \begin{pmatrix} \cancel{\hat{y}_0(t+d+1/t)} \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

Final expression

$$\begin{pmatrix} \hat{y}(t+h_i/t) \\ \hat{y}(t+h_i+1/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_{h_i-d-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_{h_p-d-h_c} \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_c-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+h_i/t) \\ \hat{y}_0(t+h_i+1/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

$$\hat{Y}(t+h_p/t) = \Phi_r D(q^{-1}) U(t+h_c-1) + \hat{Y}_0(t+h_p/t)$$

The criteria

$$\hat{J}(U(t+h_c-1)) = \sum_{j=h_i}^{h_p} \{\hat{y}(t+j/t) - y^*(t+j)\}^2 + \sum_{j=h_i}^{h_p} \lambda\{D(q^{-1})u(t+j-h_i))\}^2$$

$$= \left\| \hat{Y}\left(t+h_p/t\right) - Y^*\left(t+h_p/t\right) \right\|^2 + \lambda \| D(q^{-1})U(t+h_c-1)\|^2$$

$$with \quad \mathrm{Y}^*\left(t+h_p/t\right) = \begin{pmatrix} y^*(t+h_i/t) \\ y^*(t+h_i+1/t) \\ \vdots \\ y^*\left(t+h_p/t\right) \end{pmatrix}$$

The criteria

$$\hat{J}(U(t+h_c-1)) = \left\|\hat{Y}\left(t+h_p/t\right) - Y^*\left(t+h_p/t\right)\right\|^2 + \lambda\|D(q^{-1})U(t+h_c-1)\|^2$$

$$\frac{\partial\hat{J}(U(t+h_c-1))}{\partial(U(t+h_c-1))} = \frac{\partial\hat{J}(U(t+h_c-1))}{\partial(D(q^{-1})U(t+h_c-1))}\frac{\partial(D(q^{-1})U(t+h_c-1))}{\partial(U(t+h_c-1))}$$

$$= \frac{\partial\hat{J}(U(t+h_c-1))}{\partial(D(q^{-1})U(t+h_c-1))}$$

The criteria

$$\hat{J}(U(t+h_c-1)) = \left\|\hat{Y}(t+h_p/t) - Y^*(t+h_p/t)\right\|^2 + \lambda\|D(q^{-1})U(t+h_c-1)\|^2$$

$$= \left\|\Phi_r D(q^{-1})U(t+h_c-1) + \hat{Y}_0(t+h_p/t) - Y^*(t+h_p/t)\right\|^2 + \lambda\|D(q^{-1})U(t+h_c-1)\|^2$$

$$\frac{\partial \hat{J}(U(t+h_c-1))}{\partial(D(q^{-1})U(t+h_c-1))} = 2{\Phi_r}^T\left(\Phi_r D(q^{-1})U(t+h_c-1) + \hat{Y}_0(t+h_p/t) - Y^*(t+h_p/t)\right)$$

$$+2\lambda I D(q^{-1})U(t+h_c-1)$$

The criteria

$$\frac{\partial \hat{J}(U(t+h_c-1)\ )}{\partial (D(q^{-1})U(t+h_c-1)\ )} = 0$$

$$\left(\Phi_r{}^T\Phi_r + \lambda I\right)D(q^{-1})U(t+h_c-1) = \Phi_r{}^T\left(Y^*(t+h_p) - \hat{Y}_0(t+h_p/t)\right)$$

$$D(q^{-1})U(t+h_c-1) = \left(\Phi_r{}^T\Phi_r + \lambda I\right)^{-1}\Phi_r{}^T\left(Y^*(t+h_p) - \hat{Y}_0(t+h_p/t)\right)$$

Receding horizon concept

We only keep the first element of $D(q^{-1})U(t+h_c-1)$ *e.g* $D(q^{-1})u(t)$

Why ?

If we keep and apply $D(q^{-1})U(t+h_c-1)$ *the system operates in open loop during sampling periods*

$$D(q^{-1})U(t+h_c-1) = \left({\Phi_r}^T\Phi_r+\lambda I\right)^{-1}{\Phi_r}^T\left(Y^*(t+h_p)-\hat{Y}_0(t+h_p/t)\right)$$

$$D(q^{-1})u(t) = \sum_{j=h_i}^{h_p}\gamma_{j-h_i}\left(y^*(t+j)-\hat{y}_0(t+j/t)\right)$$

γ_j *Elements of the first line of* $\left({\Phi_r}^T\Phi_r+\lambda I\right)^{-1}{\Phi_r}^T$

Linear Time Invariant Controller

Linear Time Invariant Equivalent Controller

$$D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_{j-h_i} \left(y^*(t+j) - \hat{y}_0(t+j/t)\right)$$

Introduce the expression of $\hat{y}_0(t+j/t)$

We operate by $C(q^{-1})$

$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_{j-h_i} C(q^{-1})\left(y^*(t+j) - \hat{y}_0(t+j/t)\right)$$

$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_{j-h_i} C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_{j-h_i} C(q^{-1})\hat{y}_0(t+j/t)$$

Linear Time Invariant Equivalent Controller

$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_{j-h_i} C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_{j-h_i} C(q^{-1})\hat{y}_0(t+j/t)$$

$$C(q^{-1})\hat{y}_0(t+j/t) = F_j(q^{-1})y(t) + H_{j-d}(q^{-1})D(q^{-1})u(t-1)$$

$$\Rightarrow \quad C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_{j-h_i} C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_{j-h_i} F_j(q^{-1})y(t)$$

$$- \sum_{j=h_i}^{h_p} \gamma_{j-h_i} H_{j-d}(q^{-1})D(q^{-1})u(t-1)$$

Linear Time Invariant Equivalent Controller

$$C(q^{-1})D(q^{-1})u(t) + \sum_{j=h_i}^{h_p} \gamma_{j-h_i} H_{j-d}(q^{-1})D(q^{-1})u(t-1) + \sum_{j=h_i}^{h_p} \gamma_{j-h_i} F(q^{-1})y(t) = \sum_{j=h_i}^{h_p} \gamma_{j-h_i} C(q^{-1})y^*(t+j)$$

$$R(q^{-1})y(t) + S(q^{-1})u(t) = T(q^{-1})y^*(t+d+1)$$

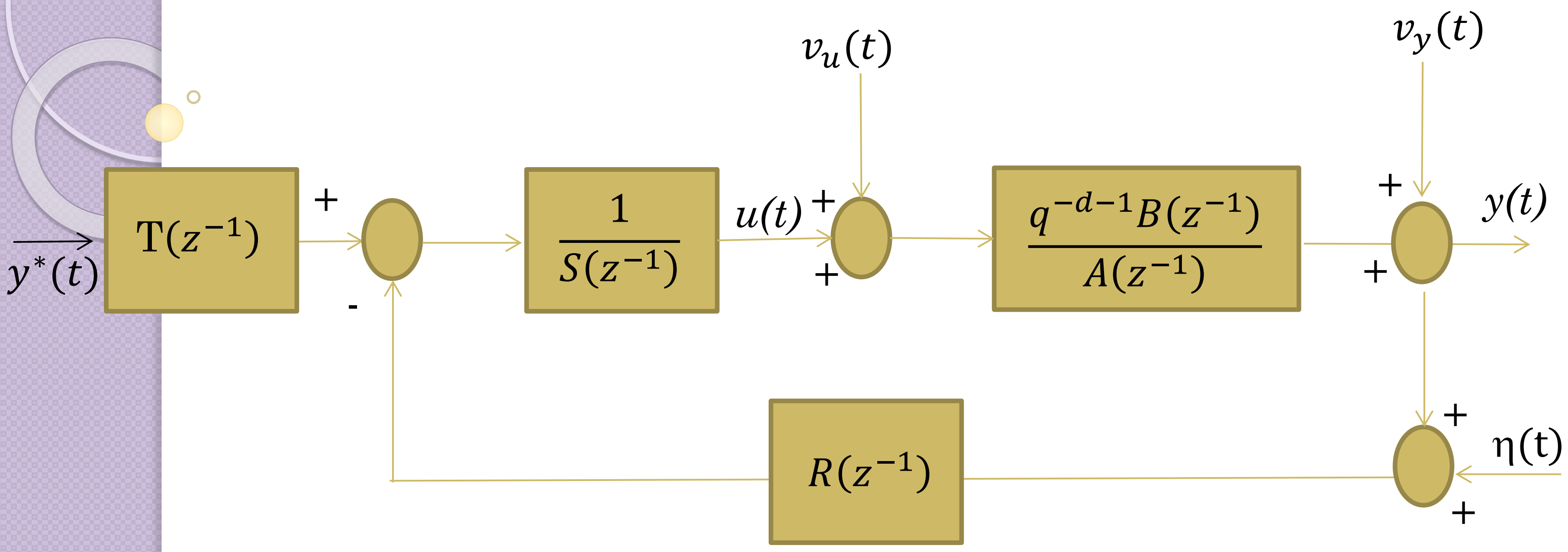
$$R(q^{-1}) = \sum_{j=h_i}^{h_p} \gamma_{j-h_i} F_j(q^{-1})$$

$$S(q^{-1}) = D(q^{-1})\left\{C(q^{-1}) + \sum_{j=h_i}^{h_p} \gamma_{j-h_i} q^{-1} H_{j-d}(q^{-1})\right\}$$

$$T(q^{-1}) = C(q^{-1}) \sum_{j=h_i}^{h_p} \gamma_{j-h_i}$$

Input - Output performances

Closed-loop performances



$v_u(t)$ *Input disturbance (low frequency)*

$v_y(t)$ *Output disturbance (low frequency)*

$y^*(t)$ *Reference sequence*

$\eta(t)$ *Noise mesurements (high frequency)*

Closed-loop performances

Output performances

$$y(t) = \frac{q^{-d-1}B(q^{-1})T(q^{-1})}{P_c(z^{-1})}y^*(t) \qquad \text{Tracking dynamics}$$

$$\left.\begin{aligned} &+\frac{A(q^{-1})S(q^{-1})}{P_c(z^{-1})}v_y(t) \\ &+\frac{B(q^{-1})S(q^{-1})}{P_c(z^{-1})}v_u(t) \\ &+\frac{S(q^{-1})}{P_c(z^{-1})}\eta(t) \end{aligned}\right\} \text{Disturbance rejection dynamics}$$

Closed-loop performances

Input performances

$$u(t) = \frac{A(q^{-1})T(q^{-1})}{P_c(z^{-1})} y^*(t) \qquad \textit{Tracking dynamics}$$

$$\left.\begin{aligned} &+ \frac{A(q^{-1})S(q^{-1})}{P_c(z^{-1})} v_y(t) \\ &+ \frac{B(q^{-1})S(q^{-1})}{P_c(z^{-1})} v_u(t) \\ &+ \frac{A(q^{-1})R(q^{-1})}{P_c(z^{-1})} \eta(t) \end{aligned}\right\} \textit{Disturbance rejection dynamics}$$

Design parameters

λ	Input weighting
h_i	Initialization Horizon
h_c	Control Horizon
h_p	Prediction Horizon
$C(q^{-1})$	Prediction dynamics
$\mathrm{D}(q^{-1})$	Disturbance Model

Key properties

$$P_c(q^{-1}) = C(q^{-1})P_f(q^{-1})$$

Separation theorem

Property 1 $h_p = h_i = d + 1, h_c = 1, \lambda = 0$

$$\rightarrow P_f(q^{-1}) = \frac{1}{b_0} B(q^{-1})$$

Minimal Variance Control
(Lecture n°2)

Property 2 $h_p = h_i = d + 1, h_c = 1, \lambda \neq 0$

One step Ahead Predictive Control
(Lecture n°2)

Key properties

Property 3 $h_i = n_b + d + 1, h_c = n_a + n_d, h_p > h_i + h_c, \lambda = 0$

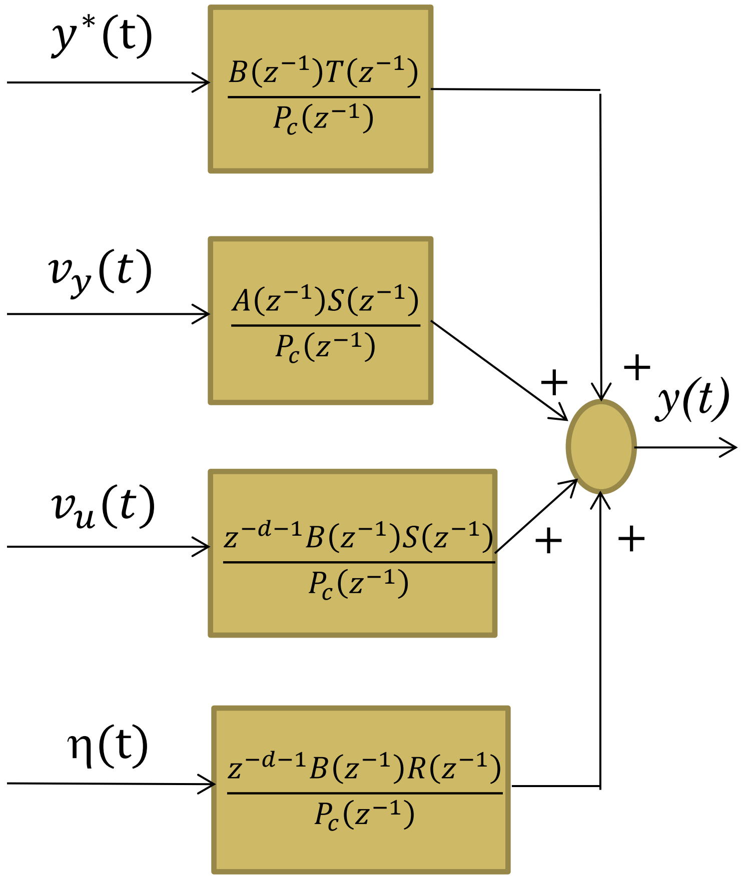
$\rightarrow P_f(q^{-1}) = 1$

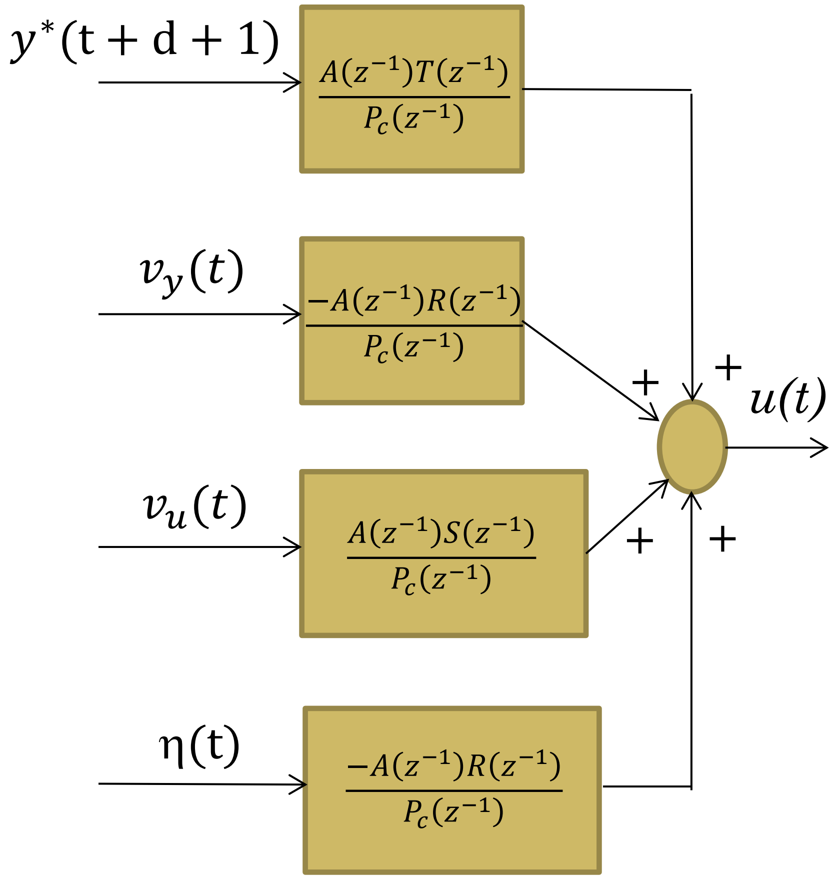
$\rightarrow P_c(q^{-1}) = C(q^{-1})$ ***Pole placement***

Property 4 $h_i = d + 1, h_c = 1, \lambda = 0, h_p \rightarrow \propto, D(q^{-1}) = 1 - q^{-1}$

$\rightarrow P_f(q^{-1}) = A(q^{-1})$ ***Internal Model Control***

Usual Sensitivity functions





Usual Sensitivity functions

Sensitivity function

$$\Psi(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{P_c(z^{-1})} = \frac{y(z^{-1})}{v_y(z^{-1})} = \frac{u(z^{-1})}{v_u(z^{-1})}$$

Complementary Sensitivity function

$$\Gamma(z^{-1}) = \frac{B(z^{-1})R(z^{-1})}{P_c(z^{-1})} = \frac{y(z^{-1})}{\eta(z^{-1})}$$

Sensitivity function × Controller

$$\Psi(z^{-1})\frac{R(z^{-1})}{S(z^{-1})} = \frac{u(z^{-1})}{\eta(z^{-1})}$$

Sensitivity function × System

$$\Psi(z^{-1})\frac{B(z^{-1})}{A(z^{-1})} = \frac{y(z^{-1})}{v_u(z^{-1})}$$

Controller Design

Parameters choice ⇨ *Shaping of the usual sensitivity functions*

Sensitivity function

High pass filter ⟶ *Disturbance rejection* $v_y(t)$
Bandwidth ⟶ *Dynamics*
Modulus Margin ⟶ *Robustness / modeling errors*

Complementary Sensitivity function

Low pass filter ⟶ *Noise rejection / y(t)*
Bandwidth ⟶ *Dynamics*

Sensitivity function × Controller

Low pass filter ⟶ *Noise rejection / u(t)*
Bandwidth ⟶ *Dynamics*

Sensitivity function × System

High pass filter ⟶ *Disturbance rejection* $v_u(t)$
Bandwidth ⟶ *Dynamics*