Predictive control

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Schedule

1 – Optimal prediction for control design purposes

2 – Criteria and derivation of the criteria

3 – Linear Time Invariant controller

4 – Input / Output performances

Optimal prediction for control design purposes

New parametrization of the predictor

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})} y(t) + \frac{B(q^{-1})E(q^{-1})}{C(q^{-1})} D(q^{-1}) u(t+j-d-1) + E(q^{-1}) \gamma(t+j)$$

Let's have a look at $\frac{B(q^{-1})E(q^{-1})}{C(q^{-1})} D(q^{-1}) u(t+j-d-1)$

It contains terms $\{u(t+j-d-1) \quad \ldots \quad u(t)\}$ *and* $\{u(t-1) \quad u(t-2) \quad ..\}$

Future control values that have to be calculated

Already available at time t

New parametrization of the predictor

We are going to separate the future and the past values of the control values

We can do that with a second polynomial division

$$\frac{B(q^{-1})E(q^{-1})}{C(q^{-1})} = G_{j-d}(q^{-1}) + q^{-j+d}\frac{H_{j-d}(q^{-1})}{C(q^{-1})}$$

$$G(q^{-1}) = g_0 + g_1 q^{-1} + \cdots g_{j-d-1} q^{-j+d+1}$$

$$H(q^{-1}) = h_0 + h_1 q^{-1} + \cdots h_{nh} q^{-nh}$$

!

The degree (j-d-1) of $G(q^{-1})$ plays an important role

New parametrization of the predictor

Hence

$$\frac{B(q^{-1})E(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1)$$

$$= G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + q^{-j+d}\frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t+j-d-1)$$

$$= \left(g_0 + g_1 q^{-1} + .. + g_{j-d-1}q^{-j+d+1}\right)D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)$$

$$= \underbrace{g_0 D(q^{-1})u(t+j-d-1) + .. + g_{j-d-1}D(q^{-1})u(t)}_{\text{Future control values (to be calculated)}} + \underbrace{\frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)}_{\text{Last control values (already known)}}$$

Future control values (to be calculated)

Last control values (already known)

New parametrization of the predictor

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})} y(t) + \frac{B(q^{-1})E(q^{-1})}{C(q^{-1})} D(q^{-1}) u(t+j-d-1) + E(q^{-1}) \gamma(t+j)$$

$$\Downarrow$$

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})} y(t) + G_{j-d}(q^{-1}) D(q^{-1}) u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})} D(q^{-1}) u(t-1) + E(q^{-1}) \gamma(t+j)$$

To summurize

$$G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1)$$

Only depends on
the actual and future control values

$$\frac{F(q^{-1})}{C(q^{-1})}y(t)+\frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)$$

Is completely known at time t

$$E(q^{-1})\gamma(t+j)$$

Is unpredictible

The optimal j-step predictor

$$y(t+j) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1) + E(q^{-1})\gamma(t+j)$$

What is avalaible for prediction ?

$$\hat{y}(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)$$

$$\hat{y}(t+j/t) = G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \hat{y}_0(t+j/t)$$

with $$\hat{y}_0(t+j/t) = \frac{F(q^{-1})}{C(q^{-1})}y(t) + \frac{H_{j-d}(q^{-1})}{C(q^{-1})}D(q^{-1})u(t-1)$$

A set of predictors

$$\hat{y}(t+j/t) = G_{j-d}(q^{-1})D(q^{-1})u(t+j-d-1) + \hat{y}_0(t+j/t)$$

From $j = d+1$ to $j = h_p$

$$\hat{y}(t+d+1/t) = g_0 D(q^{-1})u(t) + \hat{y}_0(t+d+1/t)$$

$$\hat{y}(t+d+2/t) = g_0 D(q^{-1})u(t+1) + g_1 D(q^{-1})u(t) + \hat{y}_0(t+d+2/t)$$

$$\hat{y}(t+h_p/t) = g_0 D(q^{-1})u(t+h_p-d-1) + \ldots g_{h_p-d-1} D(q^{-1})u(t) + \hat{y}_0(t+h_p/t)$$

Matricial Form of the set of predictors

$$\begin{pmatrix} \hat{y}(t+d+1/t) \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+d+1/t) \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

$$\Downarrow$$

$$\hat{Y}(t+h_p/t) = \Phi D(q^{-1}) U(t+h_p-d-1) + \hat{Y}_0(t+h_p/t)$$

Criteria and derivation of the criteria

The criteria

- *Find the control vector* $U(t+h_c-1)$ *that minimizes*

$$J(U(t+h_c-1)\ ) = \varepsilon\left(\sum_{j=h_i}^{h_p}\{y(t+j)-y^*(t+j)\}^2\right)+\varepsilon\left(\sum_{j=h_i}^{h_p}\lambda\{D(q^{-1})u(t+j-h_i))\}^2\right)$$

with $U(t+h_c-d-1) = [u(t) \quad \ldots \quad u(t+h_c-1)]$

$\mathrm{u}(t+j) = 0 \quad \forall j \geq h_c$

An equivalent criteria

- *The control vector* $U(t+h_c-1)$ *that minimizes*

$$J(U(t+h_c-1)\,)=\varepsilon\left(\sum_{j=h_i}^{h_p}\{y(t+j)-y^*(t+j)\}^2\right)+\varepsilon\left(\sum_{j=h_i}^{h_p}\lambda\{D(q^{-1})u(t+j-h_i))\}^2\right)$$

Also minimizes

$$\hat{J}(U(t+h_c-1)\,)=\sum_{j=h_i}^{h_p}\{\hat{y}(t+j/t)-y^*(t+j)\}^2+\sum_{j=h_i}^{h_p}\lambda\{D(q^{-1})u(t+j-h_i))\}^2$$

New matricial expression for the set of predictors

$$\begin{pmatrix} \hat{y}(t+d+1/t) \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+d+1/t) \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

1 *We need the predictors from* h_i *to* h_p

We suppress the first lines

$$\begin{pmatrix} \cancel{\hat{y}(t+d+1/t)} \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} \cancel{g_0} & \cancel{\cdots} & \cancel{0} \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_p-d-1) \end{pmatrix} + \begin{pmatrix} \cancel{\hat{y}_0(t+d+1/t)} \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

2 *Control values are nul if* $j \geq h_c$

We suppress the last columns of Φ

$$\begin{pmatrix} \cancel{\hat{y}(t+d+1/t)} \\ \hat{y}(t+d+2/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} \cancel{g_0 \quad \cdots \quad 0} \\ \vdots \quad \ddots \quad \vdots \\ g_{h_p-d-1} \quad \cdots \quad g_0 \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ \cancel{u(t+h_p-d-1)} \end{pmatrix} + \begin{pmatrix} \cancel{\hat{y}_0(t+d+1/t)} \\ \hat{y}_0(t+d+2/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

Final expression

$$\begin{pmatrix} \hat{y}(t+h_i/t) \\ \hat{y}(t+h_i+1/t) \\ \vdots \\ \hat{y}(t+h_p/t) \end{pmatrix} = \begin{pmatrix} g_{h_i-d-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ g_{h_p-d-1} & \cdots & g_{h_p-d-h_c} \end{pmatrix} D(q^{-1}) \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+h_c-1) \end{pmatrix} + \begin{pmatrix} \hat{y}_0(t+h_i/t) \\ \hat{y}_0(t+h_i+1/t) \\ \vdots \\ \hat{y}_0(t+h_p/t) \end{pmatrix}$$

$$\hat{Y}(t+h_p/t) = \Phi_r D(q^{-1}) U(t+h_c-1) + \hat{Y}_0(t+h_p/t)$$

The criteria

$$\hat{J}(U(t+h_c-1)\ ) = \sum_{j=h_i}^{h_p}\{\hat{y}(t+j/t) - y^*(t+j)\}^2 + \sum_{j=h_i}^{h_p}\lambda\{D(q^{-1})u(t+j-h_i))\}^2$$

$$= \left\|\hat{Y}\left(t+h_p/t\right) - Y^*\left(t+h_p/t\right)\right\|^2 + \lambda\|D(q^{-1})U(t+h_c-1)\|^2$$

$$with \quad \mathrm{Y}^*\left(t+h_p/t\right) = \begin{pmatrix} y^*(t+h_i/t) \\ y^*(t+h_i+1/t) \\ \vdots \\ y^*\left(t+h_p/t\right) \end{pmatrix}$$

The criteria

$$\hat{J}(U(t+h_c-1)) = \left\|\hat{Y}(t+h_p/t) - Y^*(t+h_p/t)\right\|^2 + \lambda\|D(q^{-1})U(t+h_c-1)\|^2$$

$$\frac{\partial\hat{J}(U(t+h_c-1))}{\partial(U(t+h_c-1))} = \frac{\partial\hat{J}(U(t+h_c-1))}{\partial(D(q^{-1})U(t+h_c-1))}\frac{\partial(D(q^{-1})U(t+h_c-1))}{\partial(U(t+h_c-1))}$$

$$= \frac{\partial\hat{J}(U(t+h_c-1))}{\partial(D(q^{-1})U(t+h_c-1))}$$

The criteria

$$\hat{J}(U(t+h_c-1)) = \left\|\hat{Y}\left(t+h_p/t\right) - Y^*\left(t+h_p/t\right)\right\|^2 + \lambda\|D(q^{-1})U(t+h_c-1)\|^2$$

$$= \left\|\Phi_r D(q^{-1})U(t+h_c-1) + \hat{Y}_0\left(t+h_p/t\right) - Y^*\left(t+h_p/t\right)\right\|^2 + \lambda\|D(q^{-1})U(t+h_c-1)\|^2$$

$$\frac{\partial \hat{J}(U(t+h_c-1))}{\partial(D(q^{-1})U(t+h_c-1))} = 2\Phi_r{}^T\left(\Phi_r D(q^{-1})U(t+h_c-1) + \hat{Y}_0\left(t+h_p/t\right) - Y^*\left(t+h_p/t\right)\right)$$
$$+2\lambda I D(q^{-1})U(t+h_c-1)$$

The criteria

$$\frac{\partial \hat{J}(U(t+h_c-1)\ )}{\partial(D(q^{-1})U(t+h_c-1)\ )} = 0$$

$$\left(\Phi_r{}^T\Phi_r + \lambda I\right)D(q^{-1})U(t+h_c-1) = \Phi_r{}^T\left(Y^*(t+h_p) - \hat{Y}_0(t+h_p/t)\right)$$

$$D(q^{-1})U(t+h_c-1) = \color{red}{\left(\Phi_r{}^T\Phi_r + \lambda I\right)^{-1}}\Phi_r{}^T\left(Y^*(t+h_p) - \hat{Y}_0(t+h_p/t)\right)$$

Receding horizon concept

We only keep the first element of $D(q^{-1})U(t+h_c-1)$ *e.g* $D(q^{-1})u(t)$

Why ?

If we keep and apply $D(q^{-1})U(t+h_c-1)$ *the system operates in open loop during sampling periods*

$$\Downarrow$$

$$D(q^{-1})U(t+h_c-1) = \left({\Phi_r}^T\Phi_r + \lambda I\right)^{-1}{\Phi_r}^T\left(Y^*(t+h_p) - \hat{Y}_0(t+h_p/t)\right)$$

$$\Downarrow$$

$$D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j\left(y^*(t+j) - \hat{y}_0(t+j/t)\right)$$

γ_j *Elements of the first line of* $\left({\Phi_r}^T\Phi_r + \lambda I\right)^{-1}{\Phi_r}^T$

Linear Time Invariant Controller

Linear Time Invariant Equivalent Controller

$$D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j \left(y^*(t+j) - \hat{y}_0(t+j/t)\right)$$

Introduce the expression of $\hat{y}_0(t+j/t)$

We operate by $C(q^{-1})$

$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j \, C(q^{-1})\left(y^*(t+j) - \hat{y}_0(t+j/t)\right)$$

$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j \, C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_j \, C(q^{-1})\hat{y}_0(t+j/t)$$

Linear Time Invariant Equivalent Controller

$$C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j \, C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_j \, C(q^{-1})\hat{y}_0(t+j/t)$$

$$C(q^{-1})\hat{y}_0(t+j/t) = F(q^{-1})y(t) + H_{j-d}(q^{-1})D(q^{-1})u(t-1)$$

$$\Rightarrow \quad C(q^{-1})D(q^{-1})u(t) = \sum_{j=h_i}^{h_p} \gamma_j \, C(q^{-1})y^*(t+j) - \sum_{j=h_i}^{h_p} \gamma_j \, F(q^{-1})y(t)$$

$$- \sum_{j=h_i}^{h_p} \gamma_j \, H_{j-d}(q^{-1})D(q^{-1})u(t-1)$$

Linear Time Invariant Equivalent Controller

$$C(q^{-1})D(q^{-1})u(t)+\sum_{j=h_i}^{h_p}\gamma_j\,H_{j-d}(q^{-1})D(q^{-1})u(t-1)+\sum_{j=h_i}^{h_p}\gamma_j\,F(q^{-1})y(t)=\sum_{j=h_i}^{h_p}\gamma_j\,C(q^{-1})y^*(t+j)$$

$$R(q^{-1})y(t)+S(q^{-1})u(t)=T(q^{-1})y^*(t+d+1)$$

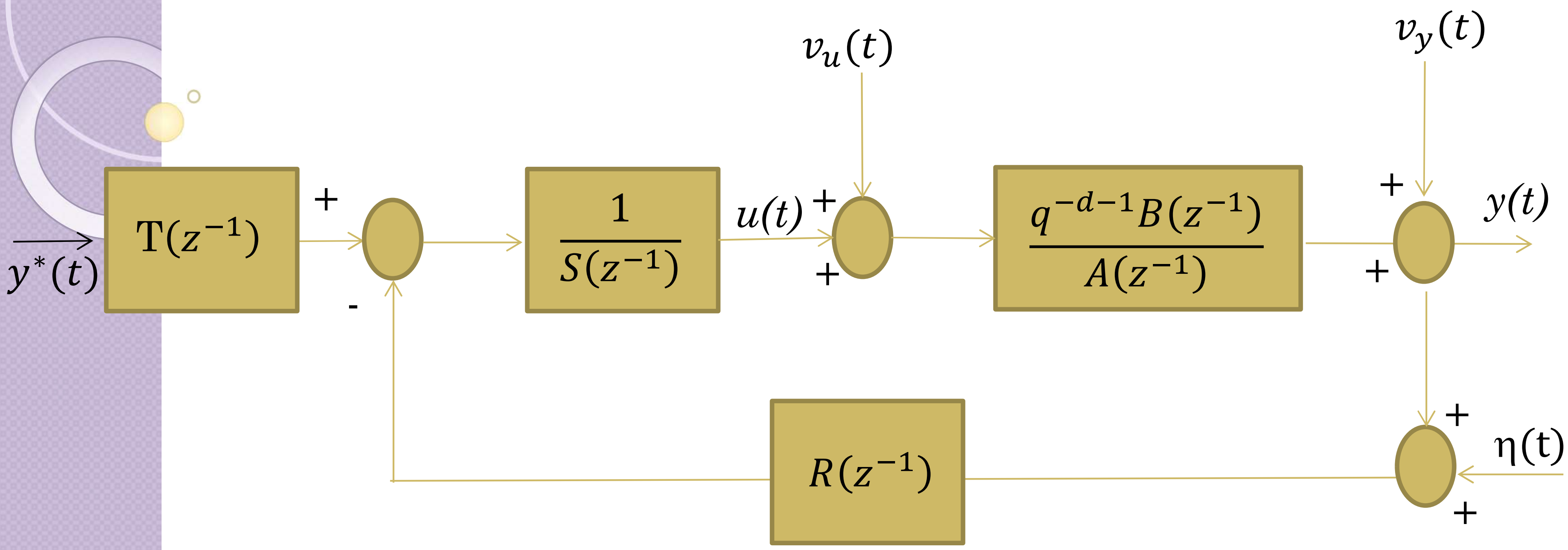
$$R(q^{-1})=\sum_{j=h_i}^{h_p}\gamma_j\,F(q^{-1})$$

$$S(q^{-1})=D(q^{-1})\left\{C(q^{-1})+\sum_{j=h_i}^{h_p}\gamma_j\,q^{-1}H_{j-d}(q^{-1})\right\}$$

$$T(q^{-1})=C(q^{-1})\sum_{j=h_i}^{h_p}\gamma_j$$

Input - Output performances

Closed-loop performances



$v_u(t)$ *Input disturbance (low frequency)*

$v_y(t)$ *Output disturbance (low frequency)*

$y^*(t)$ *Reference sequence*

$\eta(t)$ *Noise mesurements (high frequency)*

Closed-loop performances

Output performances

$$y(t) = \frac{q^{-d-1}B(q^{-1})T(q^{-1})}{P_c(z^{-1})}y^*(t) \quad \textit{Tracking dynamics}$$

$$\left.\begin{array}{l} + \dfrac{A(q^{-1})S(q^{-1})}{P_c(z^{-1})}v_y(t) \\[2ex] + \dfrac{B(q^{-1})S(q^{-1})}{P_c(z^{-1})}v_u(t) \\[2ex] + \dfrac{S(q^{-1})}{P_c(z^{-1})}\eta(t) \end{array}\right\} \textit{Disturbance rejection dynamics}$$

Input performances

$$u(t) = \frac{A(q^{-1})T(q^{-1})}{P_c(z^{-1})} y^*(t)$$

Tracking dynamics

$$+ \frac{A(q^{-1})S(q^{-1})}{P_c(z^{-1})} v_y(t) + \frac{B(q^{-1})S(q^{-1})}{P_c(z^{-1})} v_u(t) + \frac{A(q^{-1})R(q^{-1})}{P_c(z^{-1})} \eta(t)$$

Disturbance rejection dynamics

Design parameters

λ	Input weighting
h_i	Initialization Horizon
h_c	Control Horizon
h_p	Prediction Horizon
$C(q^{-1})$	Prediction dynamics
$\mathrm{D}(q^{-1})$	Disturbance Model

Key properties

$$P_c(q^{-1}) = C(q^{-1})P_f(q^{-1})$$

Separation theorem

Property 1 $h_p = h_i = d + 1, h_c = 1, \lambda = 0$

$$\rightarrow P_f(q^{-1}) = \frac{1}{b_0}B(q^{-1})$$

Minimal Variance Control
(Lecture n°2)

Property 2 $h_p = h_i = d + 1, h_c = 1, \lambda \neq 0$

One step Ahead
Predictive Control
(Lecture n°2)

Key properties

Property 3 $h_i = n_b + d + 1, h_c = n_a + n_d, h_p > h_i + h_c, \lambda = 0$

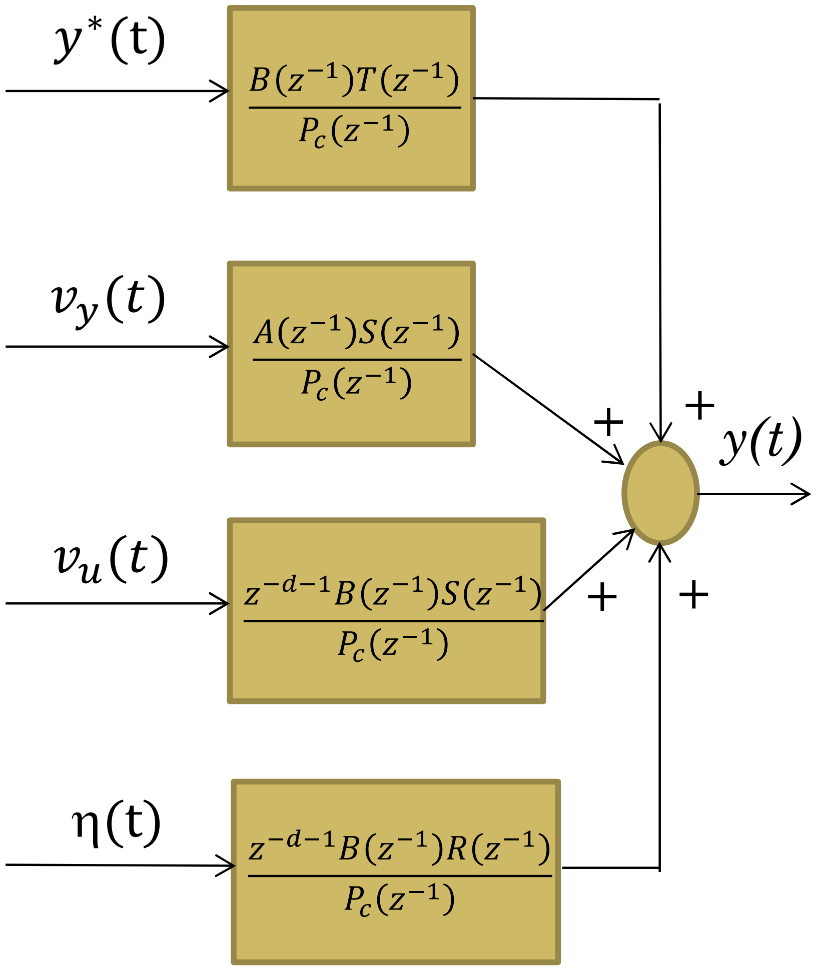
$\rightarrow P_f(q^{-1}) = 1$

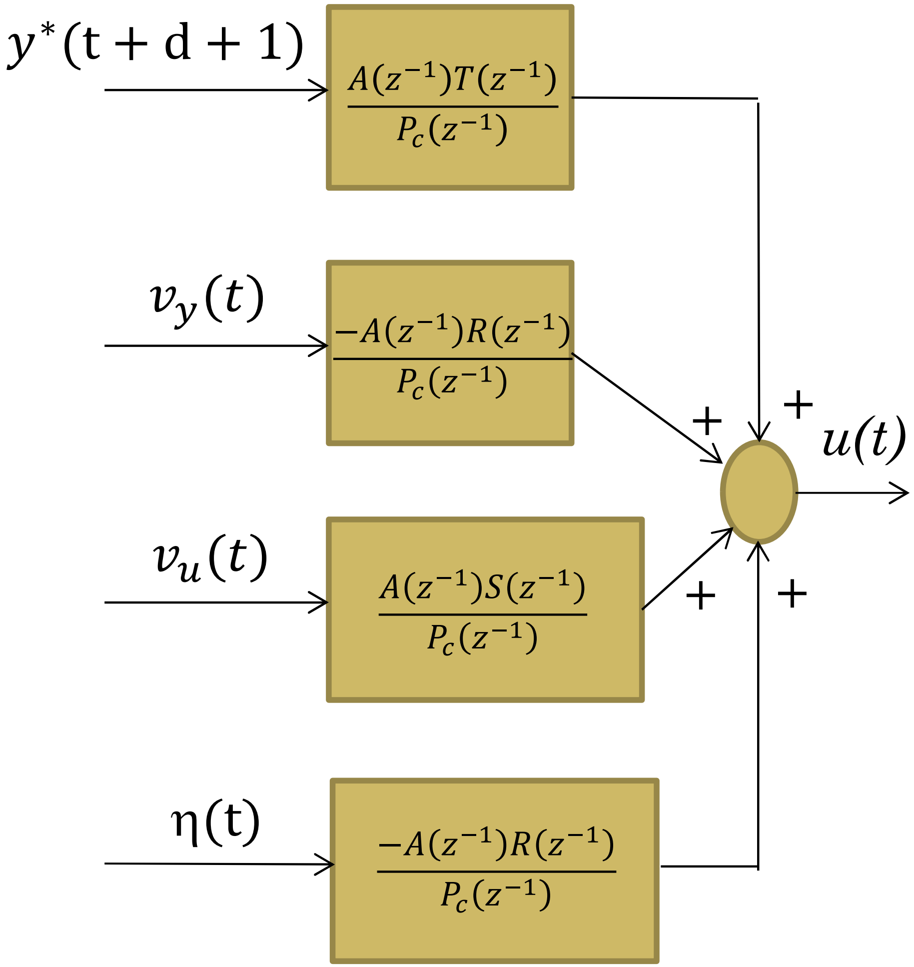
$\rightarrow P_c(q^{-1}) = C(q^{-1})$ ***Pole placement***

Property 4 $h_i = d + 1, h_c = 1, \lambda = 0, h_p \rightarrow \propto, D(q^{-1}) = 1 - q^{-1}$

$\rightarrow P_f(q^{-1}) = A(q^{-1})$ ***Internal Model Control***

Usual Sensitivity functions





Usual Sensitivity functions

Sensitivity function

$$\Psi(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{P_c(z^{-1})} = \frac{y(z^{-1})}{v_y(z^{-1})} = \frac{u(z^{-1})}{v_u(z^{-1})}$$

Complementary Sensitivity function

$$\Gamma(z^{-1}) = \frac{B(z^{-1})R(z^{-1})}{P_c(z^{-1})} = \frac{y(z^{-1})}{\eta(z^{-1})}$$

Sensitivity function × Controller

$$\Psi(z^{-1})\frac{R(z^{-1})}{S(z^{-1})} = \frac{u(z^{-1})}{\eta(z^{-1})}$$

Sensitivity function × System

$$\Psi(z^{-1})\frac{B(z^{-1})}{A(z^{-1})} = \frac{y(z^{-1})}{v_u(z^{-1})}$$

Controller Design

Parameters choice ⇨ *Shaping of the usual sensitivity functions*

Sensitivity function

High pass filter ⟶ *Disturbance rejection* $v_y(t)$
Bandwidth ⟶ *Dynamics*
Modulus Margin ⟶ *Robustness / modeling errors*

Complementary Sensitivity function

Low pass filter ⟶ *Noise rejection / y(t)*
Bandwidth ⟶ *Dynamics*

Sensitivity function × ***Controller***

Low pass filter ⟶ *Noise rejection / u(t)*
Bandwidth ⟶ *Dynamics*

Sensitivity function × ***System***

High pass filter ⟶ *Disturbance rejection* $v_u(t)$
Bandwidth ⟶ *Dynamics*